Math 280 Solutions for October 1

Pythagoras Level

#1. Suppose that $a_1, a_2, a_3, \ldots, a_n, \ldots$, is an increasing sequence of positive integers such that $a_{n+1} = a_n + a_{n1}$ for $n \geq 2$ and $a_7 = 100$. Determine the value of a_8.

[Garden State UMC 2004 #11] We have $a_3 = a_1 + a_2$, $a_4 = a_1 + 2a_2$, $a_5 = 2a_1 + 3a_2$, $a_6 = 3a_1 + 5a_2$, $a_7 = 5a_1 + 8a_2$. Now $5a_1 + 8a_2 = 100$ has only one integral solution with $a_2 > a_1 > 0$. It is $a_1 = 4$, $a_2 = 10$. Then the sequence is 4, 10, 14, 24, 38, 62, 100, 162, ...

#2. Suppose the following system of linear equations has no solutions. Find k.

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= k \\ x + y + kz &= k^2 \end{aligned}$$

[Garden State UMC 2005 Team #4] The linear system can also be written in the matrix form $AX = B$. Apply row reductions on the augment matrix we get:

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & k \\ 1 & 1 & k & k^2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & k & k^2 \\ 0 & k-1 & 1-k^2 & k-k^2 \\ 0 & 1-k & 1-k^2 & 1-k^3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & k+1 & k^2+k+1 \\ 0 & 1 & -1 & -k-1 \\ 0 & 0 & -k-2 & -k^2-2k-2 \end{array}\right]$$

Notice that if $k = 1$,the system becomes one equation $x + y + z = 1$ and hence has infinitely many solutions; so we may assume $k1 \neq 0$. For the last matrix to have no solution, $k = 2$. ($\det(A) = k^3 3k + 2 = (k1)^2(k + 2) = 0$, then A is NOT invertible.)

Newton Level

#3. Let a be a positive integer. In terms of a, determine the value of

$$A = \lim_{x \to 0} x + \cfrac{a}{x + \cfrac{a}{x + \cfrac{a}{x + \cfrac{a}{\cdots}}}}$$

[Garden State UMC 2004 #8] From the limit, we have $A = \lim_{x \to 0} x + \dfrac{a}{A}$. So $A = a/A$ and $A^2 = a$. So $A = \sqrt{a}$.

#4. Determine the numerical value of

$$\int_0^{\pi/2} \frac{\cos(x)}{\sin(x) + \cos(x)}\, dx.$$

[Garden State UMC 2005 #8] Let I denote the value of this integral. Let $u = \pi/2x$. Then and the integral becomes

$$I = \int_{\pi/2}^{0} \frac{\cos(\pi/2 - u)}{\sin(\pi/2 - u) + \cos(\pi/2 - u)}\,(-du) = \int_0^{\pi/2} \frac{\sin(u)}{\sin(u) + \cos(u)}\, dx$$

Adding the two integrals gives $2I = \int_0^{\pi/2} 1\, dx = \pi/2$, so $I = \pi/4$.

Wiles Level

#5. Let S be a set which is closed under the binary operation $\circ$ with the following properties:
(1) There is an element $e \in S$ such that $a \circ e = e \circ a = a$ for each $a \in S$.
(2) $(a \circ b) \circ (c \circ d) = (a \circ d) \circ (c \circ b)$ for all a, b, c, $d \in S$.
Prove or disprove the following statements:
(a) $\circ$ is associative on S.
(b) $\circ$ is commutative on S.

[Garden State UMC 2004 Team #5] Both hold.
(b) Commutativity:

$$\begin{aligned} a \circ b &= (e \circ a) \circ (e \circ b) \\ &= (e \circ b) \circ (e \circ a) && \text{by the given} \\ &= b \circ a. \end{aligned}$$

(a) Associativity:

$$\begin{aligned} a \circ (b \circ c) &= a \circ (c \circ b) && \text{by (b)} \\ &= (a \circ e) \circ (c \circ b) \\ &= (a \circ b) \circ (c \circ e) \\ &= (a \circ b) \circ c \end{aligned}$$

#6. Evaluate $\lim_{k\to\infty} \frac{R_k(2)}{R_k(3)}$, where

$$R_k(n) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}}}}}$$

is defined using k square-roots. Hint: Trigonometry.

[Garden State UMC 2004 Team #7] For $n = 2, 3$, we have $\cos(\pi/2n) = \sqrt{n}/2$. Then using the half-angle identity,

$$\cos(\theta) = \sqrt{\frac{\cos(2\theta) + 1}{2}}, \text{ for } 0 \le \theta < \pi/2,$$

we have $\cos(\pi/(4n)) = \frac{1}{2}\sqrt{2 + \sqrt{n}}$. In general,

$$\cos(\pi/(2^k n)) = \frac{1}{2}\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}}},$$

where there are k square roots. We can prove this by induction. Now $\sin(\theta) = \sqrt{\frac{1 - \cos(2\theta)}{2}}$, so

$$R_k(n) = \sqrt{2 - 2\cos(\pi/(2^k n))} = 2\sin(\pi/(2^{k+1} n)), \text{ for } n = 2, 3.$$

Then

$$\begin{aligned} \lim_{k\to\infty} \frac{R_k(2)}{R_k(3)} &= \lim_{k\to\infty} \frac{2\sin(\pi/(2^{k+1}2))}{2\sin(\pi/(2^{k+1}3))} \\ &= \lim_{x\to 0} \frac{\sin(x/2)}{\sin(x/3)} \\ &= \lim_{x\to 0} \frac{(1/2)\cos(x/2)}{(1/3)\cos(x/3)} \\ &= \frac{3}{2}. \end{aligned}$$