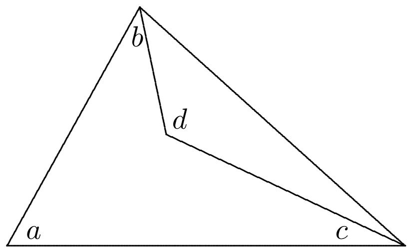
1. Angle relationship.



In the triangle at the left, a, b, c and d are measures of angles in degrees. Find a in terms of b, c and d.

2. System of equations.

Find all real solutions (x, y) of the system

$$|x| + x + y = 10,$$
$$x + |y| - y = 12.$$

Justify your answer.

3. Coefficient of x^9.

When $\left(2x^2 - \frac{1}{x^3}\right)^{12}$ is expanded and like powers of x collected, what is the coefficient of x^9?

4. Final digit.

If the number $7^{(7^7)}$ is written out in decimal form, what is the last (rightmost) digit? Defend your answer.

5. A pair of integrals.

For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx.$$

Evaluate $F(n) = I_n + I_{n+2}$.

6. Special integers.

Let a positive integer n be called "special" if the integer 1 can be expressed as a sum of n **distinct** unit fractions (i.e., fractions of the form $1/r$, where r is a positive integer). Thus, e.g., 3 is special because $1 = 1/2 + 1/3 + 1/6$. Find all special positive integers.

7. Card count.

Of two identical decks of 52 cards (it is enough to know that no two of the cards in a deck are alike), each is shuffled by itself, and one is then placed on top of the other. For each card of the top deck, one counts the number of cards lying strictly between it and the like card of the bottom deck. Find, with proof, the sum of these numbers.

8. Sum the series.

Show that

$$\sum_{n=1}^{\infty} \frac{n}{3 \cdot 5 \cdot 7 \cdots (2n+1)} = \frac{1}{2}.$$

9. Product of secants.

Prove that

$$\sec\frac{\pi}{7}\sec\frac{2\pi}{7}\sec\frac{3\pi}{7} = 8.$$

10. Powers of an irrational number.

Let $r = \sqrt{3} + \sqrt{2}$. Prove that for every positive integer n, there is a positive integer a_n satisfying

(i) $r^{2n} + r^{-2n} = 4a_n + 2$, and

(ii) $r^n = \sqrt{a_n + 1} + \sqrt{a_n}$.