Math 280 Problems for October 15

Pythagoras Level

#1. Eleven ships move bananas, lemons, and tangerines from South America to the USA. The number of bananas in each ship equals the total number of lemons on all of the remaining ships, and the number of lemons on each ship equals the total number of tangerines on all of the remaining ships. Prove that the total number of fruits on all the ships is divisible by 37.

#2. For x a real number, $\{x\}$ denotes the fractional part of x. For example, $\{5/3\} = 2/3$ and $\{3.14159\} = 0.14159$. Find, with proof, the largest real number x such that

$$\{5\{4\{3\{2\{x\}\}\}\}\} = x.$$

Newton Level

#3. If a, b, c are positive real numbers, find the value of x that minimizes the function

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b - x)^2 + c^2}.$$

(Hint: Think geometrically.)

#4. A sequence of 2×2 matrices, $\{M_n\}_{n=1}^{\infty}$, is defined as follows:

$$M_n = \begin{pmatrix} m_{11} = \frac{1}{(2n+1)!} & m_{12} = \frac{1}{(2n+2)!} \\ m_{21} = \sum_{k=0}^{n} \frac{(2n+2)!}{(2k+2)!} & m_{22} = \sum_{k=0}^{n} \frac{(2n+1)!}{(2k+1)!} \end{pmatrix}.$$

For each n, let $\det(M_n)$ denote the determinant of M_n. Determine the value of

$$\lim_{n\to\infty} \det(M_n).$$

Wiles Level

#5. Does there exist a power of 5 such that the digits of the number can be rearranged to obtain a larger power of 5? Justify your answer.

#6. The number $d_1 d_2 \ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1 d_2 \ldots d_9$ by the corresponding digit e_i ($1 \le i \le 9$) is divisible by 7. The number $f_1 f_2 \ldots f_9$ is related to $e_1 e_2 \ldots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 \ldots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]