Math 280 Solutions for October 15

Pythagoras Level

#1. Eleven ships move bananas, lemons, and tangerines from South America to the USA. The number of bananas in each ship equals the total number of lemons on all of the remaining ships, and the number of lemons on each ship equals the total number of tangerines on all of the remaining ships. Prove that the total number of fruits on all the ships is divisible by 37.

[ISMAA 2010 #2] Let B be the total number of bananas, L the total number of lemons, and T the total number of tangerines. For each ship, the number of bananas on that ship plus the number of lemons on that ship equals the total number of lemons. Adding over all ships gives

$$B + L = 11L$$

That is, $B = 10L$.
For each ship, the number of lemons on that ship plus the number of tangerines on that ship equals the total number of tangerines. Adding over all ships gives

$$L + T = 11T$$

Hence, $L = 10T$ and $B = 100T$. Therefore

$$L + B + T = 111T$$

Since 37 divides 111, the result is established.

#2. For x a real number, $\{x\}$ denotes the fractional part of x. For example, $\{5/3\} = 2/3$ and $\{3.14159\} = 0.14159$. Find, with proof, the largest real number x such that

$$\{5\{4\{3\{2\{x\}\}\}\}\} = x.$$

[ISMAA 2009 #4] First notice that if $x = n + \epsilon$ where n is an non-negative integer and $0 \le \epsilon < 1$. Then

$$\{kx\} = \{k(n+\epsilon)\} = \{kn + k\epsilon\} = \{k\epsilon\} = \{k\{x\}\}.$$

Suppose $\{5\{4\{3\{2\{x\}\}\}\}\} = x.$. Since the fractional part of a real number is non-negative, x is non-negative. Therefore, the above argument applies and $\{120x\} = x$. Let x be written in base 120. Since $x = \{120x\}$, x is less than 1 and

$$x = 0.a_1a_2a_3\ldots(\text{base } 120).$$

Now, $120x = a_1.a_2a_3\ldots$ (base 120). Thus,

$$\{120x\} = 0.a_2a_3a_4\ldots(\text{base } 120).$$

Hence, $a_1 = a_2 = a_3 = \cdots$ and all of the "digits" in the base 120 expansion of x are the same. In other words, there is an integer a between 0 and 119, inclusive, such that

$$x = \frac{a}{120} + \frac{a}{120^2} + \frac{a}{120^3} + \cdots$$

Thus,

$$x = \frac{\frac{a}{120}}{1 - \frac{1}{120}} = \frac{a}{120 - 1} = \frac{a}{119}$$

Since x must be less than 1, the largest a can be is 118. Therefore, the largest solution to the given equation is $x = 118/119$.

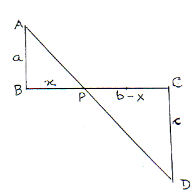
Newton Level

#3. If a, b, c are positive real numbers, find the value of x that minimizes the function

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}.$$

(Hint: Think geometrically.)

[MCMC 2010 II #2] The simplest solution is to use geometry. Consider the figure below where $AB = a$, $BC = b$, $CD = c$, and $BP = x$.



We note that $f(x) = AP + PD$, which is a minimum when P lies at the intersection of lines BC and AD. Then

$$\frac{BP}{PC} = \frac{x}{b-x} = \frac{a}{c}.$$

Hence,

$$x = \frac{ab}{a+c}.$$

#4. A sequence of 2×2 matrices, $\{M_n\}_{n=1}^{\infty}$, is defined as follows:

$$M_n = \begin{pmatrix} m_{11} = \frac{1}{(2n+1)!} & m_{12} = \frac{1}{(2n+2)!} \\ m_{21} = \sum_{k=0}^{n} \frac{(2n+2)!}{(2k+2)!} & m_{22} = \sum_{k=0}^{n} \frac{(2n+1)!}{(2k+1)!} \end{pmatrix}.$$

For each n, let $\det(M_n)$ denote the determinant of M_n. Determine the value of

$$\lim_{n\to\infty} \det(M_n).$$

[MCMC 2010 II #3]

$$\begin{aligned} \det(M_n) &= m_{11}m_{22} - m_{12}m_{21} \\ &= \sum_{k=0}^{n} \frac{1}{(2k+1)!} - \sum_{k=0}^{n} \frac{1}{(2k+2)!} \\ &= \sum_{k=1}^{2n+2} (-1)^{k+1} \frac{1}{k!} \\ &= \sum_{k=0}^{2n+2} (-1)^{k+1} \frac{1}{k!} - (-1) \\ &= 1 - \sum_{k=0}^{2n+2} (-1)^{k} \frac{1}{k!}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \det(M_n) = 1 - \lim_{n\to\infty} \sum_{k=0}^{2n+2} (-1)^k \frac{1}{k!} = 1 - e^{-1}.$$

Wiles Level

#5. Does there exist a power of 5 such that the digits of the number can be rearranged to obtain a larger power of 5? Justify your answer.

[ISMAA 2009 #6] Suppose, by way of contradiction, that such a power, say 5^k, exists. Let 5^m be the larger power of 5 obtained by rearranging the digits. Now, $k < m$. If both numbers have j digits, then

$$10^{j-1} < 5^k < 5^m < 10^j$$

Hence $5^{m-k} < 10$ and $m = k+1$. But 5^k and 5^{k+1} are assumed to have the same digits and thus, they are congruent modulo 9. That is, 9 divides $5^{k+1} - 5^k = 4 \cdot 5^k$. This is a contradiction.

#6. The number $d_1 d_2 \ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1 d_2 \ldots d_9$ by the corresponding digit e_i $(1 \le i \le 9)$ is divisible by 7. The number $f_1 f_2 \ldots f_9$ is related to $e_1 e_2 \ldots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 \ldots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

[Putnam 1995 A3] Let D and E be the numbers $d_1 \ldots d_9$ and $e_1 \ldots e_9$, respectively. We are given that $(e_i - d_i)10^{9-i} + D \equiv 0 \pmod 7$ and $(f_i - e_i)10^{9-i} + E \equiv 0 \pmod 7$ for $i = 1, \ldots, 9$. Sum the first relation over $i = 1, \ldots, 9$ and we get $E - D + 9D \equiv 0 \pmod 7$, or $E + D \equiv 0 \pmod 7$. Now add the first and second relations for any particular value of i and we get $(f_i - d_i)10^{9-i} + E + D \equiv 0 \pmod 7$. But we know $E + D$ is divisible by 7, and 10 is coprime to 7, so $d_i - f_i \equiv 0 \pmod 7$.