Learning Your ABC

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Primes!

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Examples

- $662415793599696251 = 239 \cdot 57301 \cdot 94873 \cdot 509833$
- $\dfrac{27008742384}{27680640625} = 2^4 \cdot 3^5 \cdot 5^{-6} \cdot 11^{-6} \cdot 13 \cdot 17^2 \cdot 43^2$

Factorizations of Consectutive Numbers

$$
\begin{aligned}
2 &= 2 \\
3 &= 3 \\
4 &= 2^2 \\
5 &= 5 \\
6 &= 2 \cdot 3 \\
7 &= 7 \\
8 &= 2^3 \\
9 &= 3^2 \\
10 &= 2 \cdot 5 \\
11 &= 11 \\
12 &= 2^2 \cdot 3
\end{aligned}
$$

Factorizations of Consectutive Numbers

$$
\begin{aligned}
22 &= 2 \cdot 11 \\
23 &= 23 \\
24 &= 2^3 \cdot 3 \\
25 &= 5^2 \\
26 &= 2 \cdot 23 \\
27 &= 3^3 \\
28 &= 2^2 \cdot 7 \\
29 &= 29 \\
30 &= 2 \cdot 3 \cdot 5 \\
31 &= 31 \\
32 &= 2^5
\end{aligned}
$$

Factorizations of Consectutive Numbers

$$
\begin{aligned}
122 &= 2 \cdot 61 \\
123 &= 3 \cdot 41 \\
124 &= 2^2 \cdot 31 \\
125 &= 5^3 \\
126 &= 2 \cdot 3^2 \cdot 7 \\
127 &= 127 \\
128 &= 2^7 \\
129 &= 3 \cdot 43 \\
130 &= 2 \cdot 513 \\
131 &= 131 \\
132 &= 2^2 \cdot 3 \cdot 11
\end{aligned}
$$

Factorizations of Consectutive Numbers

$$
\begin{aligned}
55122 &= 2 \cdot 3 \cdot 9187 \\
55123 &= 199 \cdot 277 \\
55124 &= 2^2 \cdot 13781 \\
55125 &= 3^2 \cdot 5^4 \cdot 7^2 \\
55126 &= 2 \cdot 43 \cdot 641 \\
55127 &= 55127 \\
55128 &= 2^3 \cdot 3 \cdot 2297 \\
55129 &= 29 \cdot 1901 \\
55130 &= 2 \cdot 5 \cdot 37 \cdot 149 \\
55131 &= 3 \cdot 17 \cdot 23 \cdot 47 \\
55132 &= 2^2 \cdot 7 \cdot 11 \cdot 179
\end{aligned}
$$

Factorizations of Consectutive Numbers

$$
\begin{aligned}
7796955122 &= 2 \cdot 11 \cdot 354407051 \\
7796955123 &= 3^2 \cdot 17 \cdot 50960491 \\
7796955124 &= 2^2 \cdot 7 \cdot 12527 \cdot 22229 \\
7796955125 &= 5^3 \cdot 62375641 \\
7796955126 &= 2 \cdot 3 \cdot 1299492521 \\
7796955127 &= 13 \cdot 23 \cdot 3929 \cdot 6637 \\
7796955128 &= 2^3 \cdot 523 \cdot 1863517 \\
7796955129 &= 3 \cdot 37 \cdot 70242839 \\
7796955130 &= 2 \cdot 5 \cdot 2777 \cdot 280769 \\
7796955131 &= 7 \cdot 229 \cdot 1487 \cdot 3271 \\
7796955132 &= 2^2 \cdot 3^3 \cdot 2503 \cdot 28843
\end{aligned}
$$

Things We Notice

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- Some rare gems are still "small" primes to a "large" power, e.g. $55125 = 3^2 \cdot 5^4 \cdot 7^2$ and $55962140625 = 3^6 \cdot 5^6 \cdot 17^3$.
- We call these numbers **smooth**.

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- If I know the prime factorization of a and b, then it's easy to find the prime factorization of ab.

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$$\begin{array}{rcccccccccc}
\frac{160000}{1058841} = a & = & 2^{8} & \cdot & 3^{-2} & \cdot & 5^{4} & \cdot & 7^{-6} & & \\
\frac{14235529}{24} = b & = & 2^{-3} & \cdot & 3^{-1} & & & \cdot & 7^{6} & \cdot & 11^{2} \\
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\frac{2420000}{27} = ab & = & & & & & & & & &
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Example

$$a + b = 2^{-3} \cdot 3^{-2} \cdot 7^{-6} \cdot 40949 \cdot 122698687$$

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Better Example: (0.96132)

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Typical Example: (0.36287)

$$7 \cdot 5701 + 37 \cdot 1361 = 2^3 \cdot 3 \cdot 3761$$

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Example: $\alpha(2, 3^{10}109, 23^5) = \dfrac{\ln(23^5)}{\ln(2 \cdot 3 \cdot 23 \cdot 109)} = 1.62991$

Good ABC Triples

- **Top three known ABC ratio (verified up to 10^{20}):**

$$\begin{aligned}(2,\ 3^{10}109,\ 23^5) &\quad \text{with} \quad \alpha = 1.62991\\ (11^2,\ 3^2 5^6 7^3,\ 2^{21}3) &\quad \text{with} \quad \alpha = 1.62599\\ (19\cdot 1307,\ 7\cdot 29^2\cdot 31^8,\ 2^8 3^{22} 5^4) &\quad \text{with} \quad \alpha = 1.62349\end{aligned}$$

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- **Largest known good ABC triples:**

$$(2^{24}5^5 47^5 181^2,\ 13^{14}19 \cdot 103 \cdot 571^2 \cdot 4261,\ 7^{28}17 \cdot 37^2)$$
$$\text{with } \alpha = 1.447420 \text{ and 29 digits}$$
$$(5^9 17^2 23^4 37^2 43 \cdot 4817,\ 3^{14}11^8 61^2 173^4,\ 2^{52}19^6 127^2)$$
$$\text{with } \alpha = 1.419184 \text{ and 28 digits}$$

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ABC Conjecture (Oesterle and Masser, 1985)

For every $\eta > 1$, there exists only a finite number of ABC triples such that

$$C > (\text{rad}(ABC))^{\eta}$$

i.e. with $\alpha(A, B, C) > \eta$.

Consequences

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There is a largest $\alpha(A, B, C)$. *It might be* 1.62991.

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Proof: Suppose there was a solution, then let $A = x^n$, $B = y^n$, $C = z^n$.
Then $\text{rad}(ABC) \leq xyz \leq z^3$. Applying the conjecture gives $z^n < (\text{rad}(ABC))^2 \leq (z^3)^2 = z^6$. Hence $n \leq 6$.
The cases of $3 \leq n \leq 6$ were proved in 1825 by Legendre and Dirichlet.

More Consequences

Corollary

If the ABC conjecture is true then the following are also proved:

- *The generalized Fermat equation*
- *Wieferich primes statement*
- *The Erdos-Woods conjecture*
- *Hall's conjecture*
- *The Erdos-Mollin-Walsh conjecture*
- *Brocard's Problem*
- *Szpiro's conjecture*
- *Mordell's conjecture*
- *Roth's theorem*
- *Dressler's conjecture*
- *Bounds for the order of the Tate-Shafarevich group*
- *Vojta's height conjecture*
- *Greenberg's conjecture*
- *The Schinzel-Tijdeman conjecture*
- *Lang's conjecture*

... and many more!

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ABC Conjecture (Rephrased)

Given $\epsilon > 0$, there exists a constant K_ϵ such that for every A, B, C coprime integers with $A + B = C$,

$$\log C \leq K_\epsilon + (1 + \epsilon) \log R$$

where $R = \text{rad}(ABC)$.

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Theorem (Gyory (2007))

Let A, B, C be coprime integers with $A + B = C$. Let t be the number of prime factors in $R = rad(ABC)$. Then

$$\log C < \frac{2^{10t+22}}{t^{t-4}} R (\log R)^t$$

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- *Example:* $x^2 + 1$ is prime, but $x^2 - 1 = (x + 1)(x - 1)$ is not. (But $x + 1$ and $x - 1$ are.)
- Let $\text{rad}(P)$ be the product of all prime polynomials dividing P.

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- A **prime polynomial** is one that cannot be factorized into smaller polynomials with rational coefficents.
- *Example:* $x^2 + 1$ is prime, but $x^2 - 1 = (x + 1)(x - 1)$ is not. (But $x + 1$ and $x - 1$ are.)
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which is just like $\ln(AB) = \ln(A) + \ln(B)$.

The PQR Theorem

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PQR Theorem (Hurwitz, Stothers, Mason)

Let P, Q, R be nonconstant relatively-prime polynomials that satisfy $P + Q = R$, then

$$\deg(R) < \deg(\text{rad}(PQR)).$$

PQR Proof

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- *Example:*

$$\begin{aligned}
F &= (x-1)^2(x^2+1)^3 \\
\text{then } F' &= 2(x-1)(x^2+1)^2(4x^2-3x+1) \\
\text{so } \gcd(F, F') &= (x-1)(x^2+1)^2 \\
\text{and } \frac{F}{\gcd(F, F')} &= (x-1)(x^2+1) = \mathrm{rad}(F).
\end{aligned}$$

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$$\begin{aligned} P + \quad Q &= \quad R \\ P' + \quad Q' &= \quad R' \end{aligned}$$

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$$\begin{aligned} {\color{red}P'}P + {\color{red}P'}Q &= {\color{red}P'}R \\ P' + Q' &= R' \end{aligned}$$

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What I Did

- The ABC Conjecture can be generalized to number fields $\mathbb{Q}(\zeta)$ where ζ is the root of a rational polynomial.

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$$\zeta^2 - \zeta - 3 = 0 \Rightarrow \zeta = \tfrac{1+\sqrt{13}}{2} \in \mathbb{Q}(\sqrt{13}),$$

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- There are "interesting" surfaces in algebraic geometry with "special" points that correspond to algebraic numbers.
- The corresponding algebraic numbers satisfy $\alpha + \beta = \gamma$ and are usually smooth.
- I used some algorithms developed in my thesis to generate 350 of these examples and computed their algebraic ABC ratios.

Results

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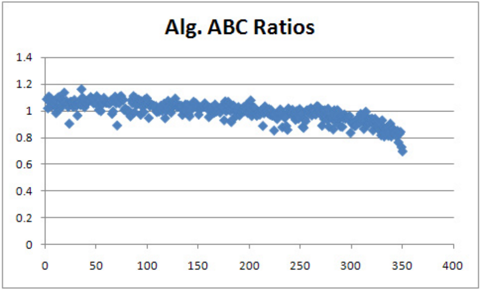

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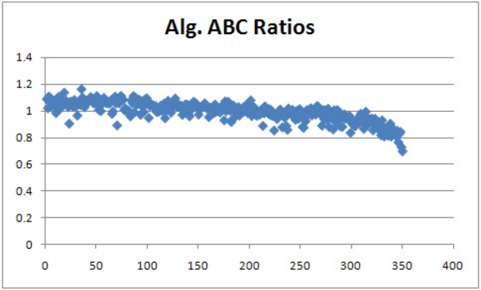

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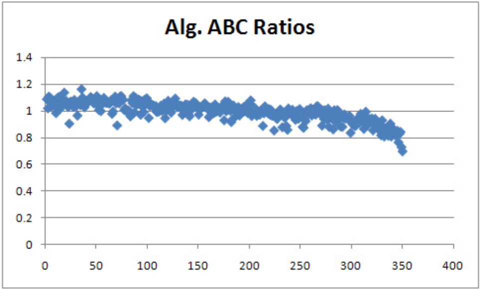

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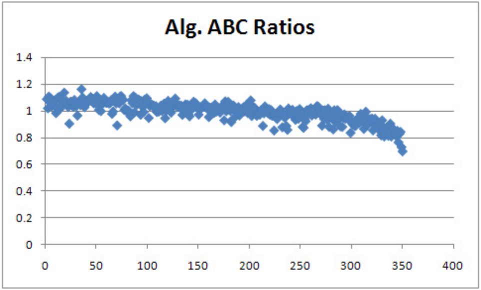

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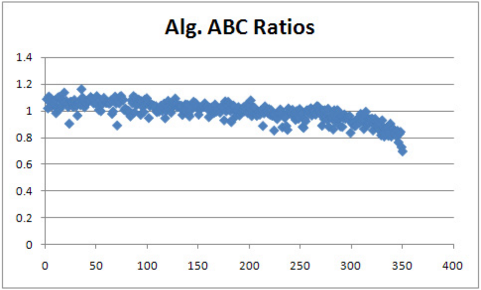

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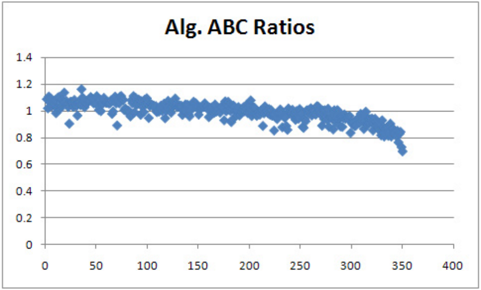

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- None of these "special" points correspond to a good ABC example.
- Data does follow a trend. Proof? No idea how to even begin.
- Failure? Well, yes, but no.

The End?

Thanks!

More information:
The ABC Conjecture Home Page
http://www.math.unicaen.fr/~nitaj/abc.html