ABSTRACT

Title of dissertation: SINGULAR MODULI OF SHIMURA CURVES

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The j-function acts as a parametrization of the classical modular curve. Its values at complex multiplication (CM) points are called singular moduli and are algebraic integers. A Shimura curve is a generalization of the modular curve and, if the Shimura curve has genus 0, a rational parameterizing function evaluated at a CM point is again algebraic over $\mathbb{Q}$. This thesis shows that the coordinate maps given in [5] for the Shimura curves associated to the quaternion algebras with discriminants 6 and 10 are Borcherds lifts of vector-valued modular forms. This property is then used to explicitly compute the rational norms of singular moduli on these curves. This method not only verifies the conjectural values for the rational CM points listed in [5], but also provides a way of algebraically calculating the norms of CM points with arbitrarily large negative discriminant.

Singular Moduli of Shimura Curves

by

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Dedication

In memory of Dad, who never stopped being proud of me.

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Chapter 1

Introduction

The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathfrak{h}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$, the union of the upper and lower half planes, via fractional linear transformations. The classical modular curve $\mathcal{X}_1^*$ is given as the one-point compactification of the Riemann surface $\mathrm{GL}_2(\mathbb{Z})\backslash\mathfrak{h}^{\pm}$. $\mathcal{X}_1^*$ is a genus-0 surface and so there exists an isomorphism $\mathcal{X}_1^* \xrightarrow{\sim} \mathbb{P}^1$. The classical choice of such a map has Fourier expansion

$$j(\tau) = \frac{1}{\mathbf{q}} + 744 + 196884\mathbf{q} + \cdots \in \frac{1}{\mathbf{q}}\mathbb{Z}[[\mathbf{q}]], \tag{1.1}$$

(where $\mathbf{q} = e^{2\pi i\tau}$) at the cusp at ∞. The j-function also provides an identification of points on the modular curve with isomorphism classes of elliptic curves. When τ is an irrational quadratic imaginary point of $\mathfrak{h}^{\pm}$, also known as a complex multiplication (CM) point, the associated elliptic curve has an extra endomorphism called a CM. When τ is a CM-point, $j(\tau)$ is called a singular modulus and is an algebraic integer. In 1984, Gross and Zagier [6] gave an explicit formula to compute the norms of singular moduli.

A Shimura curve is a generalization of the modular curve. Let B be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $D = D(B) \geq 1$ and let $\Gamma^* = N_{B^\times}(\mathcal{O}) \subset B^\times$ be the normalizer of a maximal order $\mathcal{O} \subset B$. Since there is an algebra embedding $B \hookrightarrow \mathrm{M}_2(\mathbb{R})$, the discrete group Γ^* is embedded into $\mathrm{GL}_2(\mathbb{R})$

and hence acts on $\mathfrak{h}^{\pm}$. The Shimura curve $\mathcal{X}_D^*$ is then given as

$$\mathcal{X}_D^* = \Gamma^* \backslash \mathfrak{h}^{\pm}.$$

The classical modular curve arises as the special case by compactifying when $D = 1$, $B = \mathrm{M}_2(\mathbb{Q})$ and $\mathcal{O} = \mathrm{M}_2(\mathbb{Z})$. However, when B is an indefinite division algebra, $\mathcal{X}_D^*$ is already a compact Riemann surface without cusps.

Points on a Shimura curve can also be identified with abelian varieties and again there is the notion of CM points. In addition, each CM point is associated to a quadratic imaginary field and is identified by the field discriminant of that field. As before, the image of a CM point under a rational map $\mathcal{X}_D^* \to \mathbb{P}^1$ is algebraic over $\mathbb{Q}$. However, since $\mathcal{X}_D^*$ has no cusps, such a map does not have a $\mathbf{q}$-expansion and, thus, example calculations are more difficult than in the classical case.

In [5], Elkies considered the cases of $D = 6$ and $D = 10$. First, by identifying which quadratic imaginary fields have Galois group $(\mathbb{Z}/2\mathbb{Z})^r$ for $r \leq 2$, he determined which CM points have rational coordinates on $\mathcal{X}_D^*$. Then let l be a prime with $B \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq \mathrm{M}_2(\mathbb{Q}_l)$ and define $\Gamma^*(l)$ in the classical way, i.e. $\Gamma^*(l) = \{\gamma \in \Gamma^* \mid \gamma \equiv 1 \mod l\}$. The curve $\mathcal{X}_D^*(l) = \Gamma^*(l) \backslash \mathfrak{h}^{\pm}$ is a covering of $\mathcal{X}_D^*$. Through explicit calculation of the involution on $\mathcal{X}_D^*(l)$ for small primes l, Elkies was able to compute the coordinates for about half of the rational CM points on $\mathcal{X}_6^*$ and $\mathcal{X}_{10}^*$. For example, let the coordinate map $t_6 : \mathcal{X}_6^* \to \mathbb{P}^1$ take the values of 0, 1, ∞ at $\mathcal{P}_{-4}$, $\mathcal{P}_{-24}$, $\mathcal{P}_{-3}$, the CM points of discriminant -4, -24, -3, respectively. Then using the involution on $\mathcal{X}_6^*(13)$, Elkies shows that

$$t_6(\mathcal{P}_{-312}) = \frac{7^4 23^4}{5^6 11^6}.$$

The involutions on $\mathcal{X}_D^*(l)$ for higher l are unknown and are needed to explicitly find the coordinates of the remaining half of the CM points using this method. Elkies does, however, provide a table of conjectural values for the remaining CM points obtained via numerical approximations and their behavior under standard transformations. For example, he conjectures

$$t_6(\mathcal{P}_{-163}) = \frac{3^{11}7^4 19^4 23^4}{2^{10}5^6 11^6 17^6}. \tag{1.2}$$

In this thesis, we use an alternate method that arises out of the theory of Borcherds forms to calculate the norms of singular moduli on the Shimura curves $\mathcal{X}_6^*$ and $\mathcal{X}_{10}^*$ and, as a special case, algebraically prove the conjectural values listed in [5], including (1.2). Although the methods are only demonstrated here for $D = 6$ and $D = 10$, the techniques extend to a larger class of functions $\mathcal{X}_D^* \to \mathbb{P}^1$ for arbitrary indefinite discriminants D.

Let L be a lattice in a rational inner product space $V \subset B$ with signature $(n, 2)$ and let $L^\vee$ be its dual. Then a meromorphic modular form F valued in $\mathbb{C}[L^\vee/L]$ can be given by its Fourier expansion

$$F(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\eta(m) \mathbf{q}^m e_\eta, \tag{1.3}$$

where e_η is the basis element of $\mathbb{C}[L^\vee/L]$ corresponding to η. When $c_\eta(m) \in \mathbb{Z}$ for $m < 0$, $c_0(0) = 0$, and F has weight $1 - \frac{n}{2}$, Borcherds constructs a form $\Psi(F) : \mathcal{X}_D^* \to \mathbb{P}^1$ and gives its divisor in terms of the $c_\eta(m)$ for $m < 0$ and rational quadratic divisors.

Recently, Schofer [11] provided an explicit formula in terms of the coefficients of Eisenstein series for the norm

$$\prod_{z \in Z(\Delta)} ||\Psi(z, F)||^2 \tag{1.4}$$

where $Z(\Delta)$ is the set of CM points of discriminant Δ on $\mathcal{X}_D^*$. As a corollary, he showed that since the j-function was in fact a Borcherds form, the Gross-Zagier factorization of singular moduli was a specific case of this theorem.

In the cases of $D = 6$ and $D = 10$, the coordinate map $t_D : \mathcal{X}_D^* \xrightarrow{\sim} \mathbb{P}^1$ given in [5] is defined by its divisor and normalized by its value at a chosen point. We show how this divisor can be expressed in terms of rational quadratic divisors. We then find an F_D as in (1.3) that satisfies $\text{div}(\Psi(F_D)^2) = \text{div}(t_D)$. In the cases analyzed here, the proper vector-valued F_D arises from a scalar-valued modular form that is a linear combination of Dedekind-η products. Next we compute a normalization constant, c_D, by applying (1.4) to a base case. Since the divisors are equal and the two functions agree on a chosen point,

$$\Psi(F_D)^2 = c_D t_D.$$

Finally, (1.4) can be used to calculate the norm of any CM point on $\mathcal{X}_D^*$. Specifically, we can explicitly verify (1.2). Moreover, we can compute further examples, far beyond the scope of [5]. For example, we find

$$|t_6(\mathcal{P}_{-996})| = \frac{2^{16} 7^{12} 71^4 83^2}{17^6 29^6 41^6}. \tag{1.5}$$

Chapter 2

Shimura Curves

2.1 Quaternion Algebras

In this section, we review quaternion algebras and their properties. Quaternion algebras have a long history of study, but we will mostly follow [1], [7], and [12].

Definition 2.1.1. *A quaternion algebra B over a field $\mathcal{K}$ is a central simple algebra of dimension 4 over $\mathcal{K}$.*

A quaternion algebra B over $\mathcal{K}$ is either isomorphic to $\mathrm{M}_2(\mathcal{K})$ or is a skew field. In the latter case, B is called a division $\mathcal{K}$-algebra. Let $\mathcal{R}$ be the ring of integers of $\mathcal{K}$ and let $\mathcal{K}_\nu$ denote the local completion of $\mathcal{K}$ at the place ν. Then for each place ν, $B_\nu = B \otimes_{\mathcal{K}} \mathcal{K}_\nu$ is a $\mathcal{K}_\nu$-algebra. If B_ν is a division algebra, then B is said to be ramified at ν. If B_ν is not a division algebra, then $B_\nu \simeq \mathrm{M}_2(\mathcal{K}_\nu)$. When $\mathcal{K} = \mathbb{Q}$, B is called definite (indefinite) if B ramifies (is not ramified) at the infinite place.

Definition 2.1.2. *The (reduced) discriminant $D = D(B)$ of a quaternion algebra B is given as the product of all finite ramified places of B.*

Notice that if B is isomorpic to $\mathrm{M}_2(\mathcal{K})$ then $D = \mathcal{R}$. Although the definition of D is as an ideal, when working with $\mathcal{K} = \mathbb{Q}$, D is associated with the positive generator of the principal ideal defined above. The following theorem provides a way to classify isomorphism classes of quaternion algebras.

Theorem 2.1.1 (Theorem 1.8 of [1]). *1) A quaternion algebra is ramified at a finite even number of places of $\mathcal{K}$.*

2) Given an even number of noncomplex places of $\mathcal{K}$, there exists a quaternion algebra over $\mathcal{K}$ ramified exactly at those places.

3) Two quaternion algebras are isomorphic if and only if they have the same discriminant.

When $\mathcal{K}$ has characteristic different than 2, there exists a $\mathcal{K}$-basis for B, $\{1, \alpha, \beta, \alpha\beta\}$, satisfying $\alpha\beta = -\beta\alpha$ and $\alpha^2 = a$, $\beta^2 = b$ for some $a, b \in \mathcal{K}^\times$. In this case, denote $B = \left(\frac{a,b}{\mathcal{K}}\right)$. Note that different values of a and b may produce isomorphic quaternion algebras. For example,

$$\mathrm{M}_2(\mathbb{Q}) \simeq \left(\frac{1,-1}{\mathbb{Q}}\right) \simeq \left(\frac{1,b}{\mathbb{Q}}\right) \simeq \left(\frac{a,-a}{\mathbb{Q}}\right) \simeq \left(\frac{a,1-a}{\mathbb{Q}}\right) \tag{2.1}$$

for $a, b \in \mathbb{Q}^\times$, $a \neq 1$.

Proposition 2.1.2 (Proposition 3.1 of [7]). *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ with $D = p_1 \cdots p_{2r}$. Choose q to be a prime such that $q \equiv 5 \mod 8$ and $(\frac{q}{p_i}) = -1$ for every $p_i > 2$. Then $B \simeq \left(\frac{q,D}{\mathbb{Q}}\right)$.*

There are many ways to embed $B = \left(\frac{a,b}{\mathcal{K}}\right)$ into a matrix algebra over an extension of $\mathcal{K}$. The one that will be primarily used in this thesis is

$$\phi_b : B \hookrightarrow \mathrm{M}_2(\mathcal{K}(\sqrt{b}))$$

given by

$$\phi_b(\alpha) = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

$$\phi_b(\beta) = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}. \tag{2.3}$$

There is a natural involution on $x = x_0 + x_1\alpha + x_2\beta + x_3\alpha\beta$ given by

$$\overline{x} = x_0 - x_1\alpha - x_2\beta - x_3\alpha\beta.$$

This involution allows one to define the (reduced) trace and (reduced) norm as

$$\mathrm{tr}(x) = x + \overline{x} = 2x_0, \tag{2.4}$$

$$\mathrm{n}(x) = x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2. \tag{2.5}$$

Under the above embedding, these are just the usual matrix trace and determinant.

2.2 Maximal Orders

Definition 2.2.1. *An $\mathcal{R}$-order $\mathcal{O}$ in a quaternion algebra B over $\mathcal{K}$ is an $\mathcal{R}$-ideal that is a ring. Equivalently, an $\mathcal{R}$-order $\mathcal{O}$ is a ring whose elements have trace and norm in $\mathcal{R} \subset \mathcal{O}$, and $\mathcal{O} \otimes_{\mathcal{R}} \mathcal{K} = B$. A maximal order is an order that can not be properly contained in another order.*

Example. *The case $\mathcal{H} = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ is the classical Hamiltonians and $\mathcal{H}$ contains the order $\mathbb{Z}[1, \alpha, \beta, \alpha\beta]$. However, this order is not maximal. It is contained in the maximal order $\mathbb{Z}[1, \alpha, \beta, (1 + \alpha + \beta + \alpha\beta)/2]$.*

In general, B does not have a unique maximal order. In fact, if $\omega \in B^\times$ and $\mathcal{O}$ is a maximal order, then $\omega\mathcal{O}\omega^{-1}$ is also a maximal order. However, when B is indefinite, the conjugacy class of maximal orders is unique. For example, when $B \simeq \mathrm{M}_2(\mathcal{K})$, every maximal order is conjugate to $\mathrm{M}_2(\mathcal{R})$.

Proposition 2.2.1 (Proposition 3.2 of [7]). *For B as in Proposition 2.1.2 with $\alpha^2 = q$ and $\beta^2 = D$, every maximal order is conjugate to*

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_1e_2 \tag{2.6}$$

where

$$e_1 = \frac{1+\alpha}{2}, \tag{2.7}$$

$$e_2 = \frac{m\alpha + \alpha\beta}{q}, \tag{2.8}$$

$$D \equiv m^2 \mod q. \tag{2.9}$$

Without loss of generality, we will further assume that $2D \mid m$.

When p is a ramified prime, there is a unique maximal order $\mathcal{O}_p \subset B_p$ and it is given by

$$\mathcal{O}_p = \{\omega \in B \mid (\mathrm{ord}_p \circ \mathrm{n})(\omega) \geq 0\}. \tag{2.10}$$

Hence its group of units is given by

$$\mathcal{O}_p^\times = \{\omega \in B^\times \mid (\mathrm{ord}_p \circ \mathrm{n})(\omega) = 0\}. \tag{2.11}$$

Moreover, one can choose a uniformizer $\pi_p \in B_p^\times$ such that $B_p^\times = \mathcal{O}_p^\times \rtimes \pi_p^{\mathbb{Z}}$ with $(\mathrm{ord}_p \circ \mathrm{n})(\pi_p) = 1$ and $\pi_p^2 = p$.

Define the normalizer of an order as

$$\mathrm{N}_{B^\times}(\mathcal{O}) = \{\omega \in B^\times \mid \omega\mathcal{O}\omega^{-1} \subset \mathcal{O}\}. \tag{2.12}$$

Note that in the local case where $\mathcal{O}_p$ is the unique maximal order, $\mathrm{N}_{B_p^\times}(\mathcal{O}_p) = B_p^\times$. In general, the units of an order $\mathcal{O}$ are a subgroup of $\mathrm{N}_{B^\times}(\mathcal{O})$. They are related by the following lemma, where $d(B)$ denotes the number of ramified primes of B.

Lemma 2.2.2 ([12]). $N_{B^\times}(\mathcal{O})/(\mathbb{Q}^\times\mathcal{O}^\times) \simeq (\mathbb{Z}/2\mathbb{Z})^{d(B)}$.

2.3 Shimura Curves and CM Points

From now on let $B = \left(\frac{q,D}{\mathbb{Q}}\right)$ with $\alpha^2 = q$ and $\beta^2 = D$ as in Proposition 2.1.2. Fix the embedding of $B \hookrightarrow \mathrm{M}_2(\mathbb{R})$ given by ϕ_D and the maximal order $\mathcal{O}$ as in Proposition 2.2.1. Define the following subgroups of $B^\times$,

$$\Gamma = \mathcal{O}^\times, \tag{2.13}$$

$$\Gamma^* = \mathrm{N}_{B^\times}(\mathcal{O}). \tag{2.14}$$

Their images under ϕ_D are discrete subgroups of $B^\times \subset \mathrm{GL}_2(\mathbb{R})$, and they act on $\mathfrak{h}^\pm = \mathbb{P}(\mathbb{C}) - \mathbb{P}(\mathbb{R})$ via fractional linear transformations. Define $\mathcal{X}$ and $\mathcal{X}^*$ to be the Shimura curves

$$\mathcal{X} = \mathcal{X}_D = \Gamma\backslash\mathfrak{h}^\pm, \quad \mathcal{X}^* = \mathcal{X}_D^* = \Gamma^*\backslash\mathfrak{h}^\pm. \tag{2.15}$$

When B is an indefinite division algebra, $\mathcal{X}$ and $\mathcal{X}^*$ are compact Riemann surfaces with no cusps. Also, Lemma 2.2.2 implies that $\mathcal{X}$ is a covering space of $\mathcal{X}^*$ of degree $2^{d(B)}$.

Fix a quadratic imaginary field $\boldsymbol{k}$ such that if $p \mid D$ then p does not split in $\boldsymbol{k}$. Then there are many embeddings $\iota : \boldsymbol{k} \hookrightarrow B$. However, all of the embeddings are conjugate to each other [12], i.e. for any ι_1, ι_2, there exists $\omega_{12} \in B^\times$ such that $\iota_2 = Ad(\omega_{12}) \circ \iota_1$.

Definition 2.3.1. *The image $\iota(\boldsymbol{k}^\times) \to B^\times/\mathbb{Q}^\times \subset PGL_2(\mathbb{R})$ has a unique fixed point on $\mathfrak{h}^\pm$. A complex-multiplication (CM) point of $\mathcal{X}$ (resp., $\mathcal{X}^*$) is the Γ-orbit (Γ^*-orbit) of such a point. It is said to have discriminant equal to the field discriminant of $\boldsymbol{k}$.*

Since all embeddings are conjugate, a CM point is independent of the embedding. In the classical case of $B = \mathrm{M}_2(\mathbb{Q})$, the CM points are irrational imaginary solutions to integral quadratic equations with the corresponding discriminant.

2.4 Involutions on $\mathcal{X}_D^*(l)$

In this section, we summarize the method used in [5] to calculate the coordinates of rational CM points on $\mathcal{X}^*$. The first proposition gives a way of determining which CM points should have rational coordinates. Let $\mathcal{P}_\Delta$ be the CM point with discriminant $\Delta < 0$ and let $R \subset \boldsymbol{k}$ be the maximal order in the quadratic imaginary field of discriminant Δ.

Proposition 2.4.1 ([5]). *$\mathcal{P}_\Delta$ is a rational point on $\mathcal{X}_D^*$ if and only if the class group of $\boldsymbol{k}$ is generated by ideals $I \subset R$ such that $I^2 = (p)$ for some $p \mid D$.*

This implies that for a rational CM point, the class group of $\boldsymbol{k}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ where $r \leq d(B)$. In the case of $d(B) = 2$, all such fields are known, and thus the rational CM points can be identified. (See Table 8.1 for $D = 6$ and Table 8.3 for $D = 10$.)

Now let l be a prime not dividing D, so that $B \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq \mathrm{M}_2(\mathbb{Q}_l)$. Define

$$\Gamma^*(l) = \{\gamma \in \Gamma^* \mid \gamma \equiv \pm 1 \mod l\}$$

and the congruence subgroup $\Gamma_0^*(l)$ in the same fashion as its classical counterpart. Then the curves

$$\mathcal{X}_D^*(l) = \Gamma^*(l)\backslash\mathfrak{h}^{\pm}, \qquad \mathcal{X}_{D,0}^*(l) = \Gamma_0^*(l)\backslash\mathfrak{h}^{\pm} \tag{2.16}$$

are coverings of $\mathcal{X}_D^*$ whose points are also associated to abelian varieties. From the geometric structure, $\mathcal{X}_{D,0}^*(l)$ inherits an involution $w_l : \mathcal{X}_{D,0}^*(l) \to \mathcal{X}_{D,0}^*(l)$ which preserves the set of rational CM points.

In the case of $D = 6$, the image of $\Gamma^* \hookrightarrow \mathrm{PGL}_2(\mathbb{R})$ is generated by three elements and is called a triangle group. Any coordinate map $t_6 : \mathcal{X}_6^* \to \mathbb{P}^1$ is only defined up to a $\mathrm{PGL}_2(\mathbb{R})$ action, so such a map is only well-defined once its values at three points have been given. Since there are three distinguished elements of Γ^*, let the coordinate map take the values of 0, 1, ∞ at $\mathcal{P}_{-4}$, $\mathcal{P}_{-24}$, $\mathcal{P}_{-3}$, the CM points associated to the three generators.

The covering curves $\mathcal{X}_{6,0}^*(l)$, for $l = 5, 7, 13$ have genus 0 and w_l can be expressed explicitly as a rational function. Then by examining the fixed points of w_l and the w_l-orbits of 0, 1, and ∞, Elkies was able to compute the coordinates of the CM points with discriminants $-3 \cdot 5^2$, $-4 \cdot 5^2$, $-3 \cdot 7^2$, -43, $-8 \cdot 11$, and $-cl$ where $c \mid 24$.

In order to compute the remaining ten rational CM points using this method, involutions on $\mathcal{X}_{6,0}^{*}(l)$ for higher l are needed. However, these curves have genus greater than 0 and explicit expressions for w_l are unknown. Instead, Elkies used numerical techniques to calculate the coordinates to an arbitrary precision. He then recognized them as fractional values through continued fractions and their behavior under standard transformations. For example, one expects that the factorizations of both $t_6(\mathcal{P}_\Delta)$ and $t_6(\mathcal{P}_\Delta) - 1$ should only contain small primes to large powers.

Chapter 3

Quadratic Spaces and Lattices

For a given indefinite quaternion algebra B, define the $\mathbb{Q}$-vector space

$$V = \{x \in B \mid \mathrm{tr}(x) = 0\}. \tag{3.1}$$

There is a natural quadratic form on V given by $Q(x) = \mathrm{n}(x) = -x^2$. The associated inner product is

$$\begin{aligned}(x, y) &= Q(x + y) - Q(x) - Q(y) & (3.2)\\ &= \mathrm{tr}(x\overline{y}) & (3.3)\end{aligned}$$

and has signature $(1, 2)$.

3.1 The Lattice $\mathcal{O} \cap V$

Define the lattice $L = \mathcal{O} \cap V$. Let $L^\vee$ be the dual of L given by

$$L^\vee = \{x \in V \mid (x, y) \in \mathbb{Z} \ \forall \ y \in L\}.$$

Consider $L_p^\vee / L_p$ where $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

For $p \nmid D$ and p odd, there is an isomorphism $B_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$ such that $\mathcal{O}_p \simeq \mathrm{M}_2(\mathbb{Z}_p)$. Then L_p is the set of trace zero elements of $\mathrm{M}_2(\mathbb{Z}_p)$ and $L_p^\vee / L_p$ is trivial. Thus

$$L^\vee / L \simeq \prod_{p \mid 2D} L_p^\vee / L_p. \tag{3.4}$$

Now consider $p \mid D$ and p odd. Let $\mathbb{Z}_{p^2}$ be the ring of integers in the unramified quadratic extension of $\mathbb{Q}_p$ with Galois automorphism σ. Then $\mathcal{O}_p$ can be written as

$$\mathcal{O}_p = \mathbb{Z}_{p^2} + \mathbb{Z}_{p^2}\pi_p, \tag{3.5}$$

where $\pi_p^2 = p$ and $\pi_p\alpha = \alpha^\sigma \pi_p$ for $\alpha \in \mathbb{Z}_{p^2}$. Further, $\mathbb{Z}_{p^2}$ can be written as

$$\mathbb{Z}_{p^2} = \mathbb{Z}_p + \mathbb{Z}_p\delta, \qquad \delta \notin \mathbb{Z}_p^\times, \qquad \delta^2 \in \mathbb{Z}_p^\times. \tag{3.6}$$

Then L_p is given as

$$L_p = \mathbb{Z}_p\delta + \mathbb{Z}_p\pi_p + \mathbb{Z}_p\delta\pi_p. \tag{3.7}$$

Let $x = x_1\delta + x_2\pi_p + x_3\delta\pi_p$ and $y = y_1\delta + y_2\pi_p + y_3\delta\pi_p$. Then

$$(x, y) = 2\delta^2 x_1 y_1 + 2p x_2 y_2 - 2p\delta^2 x_3 y_3, \tag{3.8}$$

giving

$$L_p^\vee = \mathbb{Z}_p\delta + p^{-1}\mathbb{Z}_{p^2}\pi_p. \tag{3.9}$$

Since $\mathbb{Z}_{p^2}/p\mathbb{Z}_{p^2} \simeq \mathbb{F}_{p^2}$, there is an isomorphism

$$\mathbb{F}_{p^2} \xrightarrow{\sim} L_p^\vee/L_p, \qquad \tilde{v} \mapsto v\pi_p^{-1} + L_p. \tag{3.10}$$

Under this isomorphism, the quadratic form Q induces the function

$$Q(\tilde{v}) = vv^\sigma p^{-1} \mod \mathbb{Z}_p, \tag{3.11}$$

which is equivalent to the norm map $\mathrm{n} : \mathbb{F}_{p^2} \to \mathbb{F}_p$ via $\mathbb{F}_p \xrightarrow{\sim} p^{-1}\mathbb{Z}_p/\mathbb{Z}_p$.

The case of $p = 2$ is very similar. Again $\mathcal{O}_2 = \mathbb{Z}_4 + \mathbb{Z}_4\pi_2$. But now write $\mathbb{Z}_4 = \mathbb{Z}_2 + \mathbb{Z}_2\delta$ with $\delta = \frac{1}{2}(1+\sqrt{5})$. Then $L_2 = \mathbb{Z}_2\sqrt{5} + \mathbb{Z}_4\pi_2$ and $L_2^\vee = \frac{1}{2}L_2$. This time the isomorphism is

$$\mathbb{F}_2 \oplus \mathbb{F}_4 \xrightarrow{\sim} L_2^\vee / L_2, \qquad (\tilde{w}, \tilde{v}) \mapsto w\tfrac{\sqrt{5}}{2} + v\pi_2^{-1} + L_2, \tag{3.12}$$

and Q induces the function

$$Q(\tilde{w}, \tilde{v}) = -\frac{1}{4}w^2 - \frac{1}{2}\mathrm{n}(v) \mod \mathbb{Z}_2. \tag{3.13}$$

This surjects onto $\frac{1}{4}\mathbb{Z}/\mathbb{Z}$, given by whether or not each of the components is nonzero.

Proposition 3.1.1. *Let D_0 be the odd part of D. Then*

$$|L^\vee / L| = 8(D_0)^2. \tag{3.14}$$

Proposition 3.1.2. *Let $B_p^\times$ act on $L_p^\vee / L_p$ via conjugation. Then the $B_p^\times$ orbits of $L_p^\vee / L_p$ for odd $p \mid D$ (resp., $p = 2$) are indexed by elements of $\mathbb{F}_p$ ($\mathbb{F}_4$).*

Proof. For odd p, write $B_p^\times$ as

$$B_p^\times = (\mathcal{O}_p^\times \cup \mathcal{O}_p^\times \pi_p)p^{\mathbb{Z}}. \tag{3.15}$$

First, the powers of p are central and hence act trivially. Then by (3.7) and (3.9)

$$L_p^\vee / L_p \xrightarrow{\sim} \mathcal{O}_p / \pi_p \mathcal{O}_p. \tag{3.16}$$

Thus the elements of $\mathcal{O}_p^\times$ act through their image under the reduction map $\mathcal{O}_p \to \mathbb{F}_{p^2}$. More explicitly, $\tilde{v} \in \mathbb{F}_{p^2}^\times$ acts via left multiplication by v/v^σ. However, this is just the action of $\mathbb{F}_{p^2}^1 = \ker(\mathrm{n} : \mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times)$. Lastly, π_p acts by σ, and so there is a surjection

$$B_p^\times \twoheadrightarrow \mathbb{F}_{p^2}^1 \rtimes < \sigma > . \tag{3.17}$$

Hence the orbits of $B_p^\times$ are indexed by the elements of $\mathbb{F}_p$.

For $p = 2$, the action of $B_2^\times$ preserves the first component of (3.12) and acts on the second component the same way it did in the odd p case. So again the orbits are indexed by the four values of Q. $\square$

3.2 The Order of the Orbits of Γ^*

Define the set

$$V(t) = \{x \in V | Q(x) = t\} \tag{3.18}$$

and $L(t) = L \cap V(t)$. The discrete groups Γ and Γ^* both act on L by conjugation, and the number of Γ^*-orbits in $L(t)$ will play an important role in Sections 7.1 and 7.2.

Let $0 > \Delta \in \mathbb{Z}$ be the field discriminant of $\boldsymbol{k} = \mathbb{Q}(\sqrt{-t})$, and set $-4t = n^2\Delta$. Then the order $\mathbb{Z}[\sqrt{-t}]$ has discriminant $-4t$. Hence, its conductor is n, and if any other order R in $\boldsymbol{k}$ contains $\mathbb{Z}[\sqrt{-t}]$, then the conductor of R divides n.

Set

$$\mathcal{E} = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(\boldsymbol{k}, B). \tag{3.19}$$

Assume that for every prime $p \mid D$, p is nonsplit in $\boldsymbol{k}$ so that $\mathcal{E}$ is nontrivial. For every $x \in L(t)$, define $\iota_x \in \mathcal{E}$ by $\iota_x(\sqrt{-t}) = x$. Notice that for $x \in L(t)$ and $a \in \mathbb{Q}^\times$, $ax \in V(a^2t)$ and thus ι_{ax} is defined by $\iota_{ax}(\sqrt{-t}) = ax = a\iota_x(\sqrt{-t})$. Hence $\iota_{ax} = a\iota_x$.

For $\iota \in \mathcal{E}$, $\iota^{-1}(\mathcal{O} \cap \iota(\boldsymbol{k}))$ is an order in $\boldsymbol{k}$. Let $\mathrm{cond}(\iota)$ denote the conductor of this order and define

$$\mathcal{E}(c) = \{\iota \in \mathcal{E} \mid \mathrm{cond}(\iota) = c\}. \tag{3.20}$$

For $x \in L$, define $\mathrm{cond}(x) = \mathrm{cond}(\iota_x)$ and let

$$L(t, c) = \{x \in L(t) \mid \mathrm{cond}(x) = c\}. \tag{3.21}$$

Note that $L(t, c) = \varnothing$ when $\gcd(c, D) \neq 1$.

Proposition 3.2.1. *For $-4t = n^2\Delta$ as above,*

$$L(t) = \coprod_{c|n} L(t, c). \tag{3.22}$$

Proof. Let $x \in L(t)$. Then $\iota_x^{-1}(\mathcal{O} \cap \iota_x(\boldsymbol{k}))$ is an order containing $\mathbb{Z}[\sqrt{-t}]$. Hence its conductor $\mathrm{cond}(x)$ must divide n. □

Proposition 3.2.2. *For any fixed t and c, there is a bijection*

$$L(t, c) \xrightarrow{\sim} \mathcal{E}(c), \qquad x \mapsto \iota_x. \tag{3.23}$$

Proof. Injectivity follows from evaluating ι_x at $\sqrt{-t}$. To prove surjectivity, suppose $\iota \in \mathcal{E}(c)$. Then $\iota(\sqrt{-t}) = x \in B$. Thus $Q(x) = -x^2 = t$, so $x \in V(t)$ and $\iota = \iota_x$. By definition, $\mathrm{cond}(x) = \mathrm{cond}(\iota) = c$. Further, since $c \mid n$, $\sqrt{-t} \in \mathbb{Z}[\sqrt{-t}] \subset \iota^{-1}(\mathcal{O} \cap \iota(\boldsymbol{k}))$. Then $\iota(\sqrt{-t}) = x \in \mathcal{O}$, so $x \in L(t)$. □

Proposition 3.2.3. *For $\iota \in \mathcal{E}$, $\iota(\sqrt{-t}) \in \mathcal{O}$ if and only if $\mathrm{cond}(\iota) \mid n$.*

Proof. $\iota(\sqrt{-t}) \in \mathcal{O}$ if and only if $\iota(\mathbb{Z}[\sqrt{-t}]) \subset \mathcal{O}$. This is if and only if $\iota^{-1}(\mathcal{O} \cap \iota(\boldsymbol{k}))$ contains $\mathbb{Z}[\sqrt{-t}]$. Thus $\mathrm{cond}(\iota) \mid n$. □

Proposition 3.2.4. Γ^* *acts on* $L(t,c)$ *via conjugation.*

Proof. Let $x \in L(t,c)$ and $\gamma \in \Gamma^*$. Then

$$\begin{aligned} Q(\gamma x \gamma^{-1}) &= -(\gamma x \gamma^{-1})^2 \\ &= -(\gamma x^2 \gamma^{-1}) \\ &= \gamma(-x^2)\gamma^{-1} \\ &= \gamma Q(x) \gamma^{-1} \\ &= Q(x). \end{aligned}$$

So $\gamma x \gamma^{-1} \in L(t)$. Now consider $y \in \iota_x^{-1}(\mathcal{O} \cap \iota_x(\boldsymbol{k}))$. Then $\iota_x(y) \in \mathcal{O} \cap \iota_x(\boldsymbol{k})$. So $\gamma \iota_x(y) \gamma^{-1} \in \gamma \mathcal{O} \gamma^{-1} \cap \gamma \iota_x(\boldsymbol{k}) \gamma^{-1}$. Since $\gamma \iota_x(y) \gamma^{-1} = \iota_{\gamma x \gamma^{-1}}(y)$ and $\Gamma^* = \mathrm{N}_{B^\times}(\mathcal{O})$, then $\iota_{\gamma x \gamma^{-1}}(y) \in \mathcal{O} \cap \iota_{\gamma x \gamma^{-1}}(\boldsymbol{k})$. Hence $y \in \iota_{\gamma x \gamma^{-1}}^{-1}(\mathcal{O} \cap \iota_{\gamma x \gamma^{-1}}(\boldsymbol{k}))$ and $\iota_x^{-1}(\mathcal{O} \cap \iota_x(\boldsymbol{k})) \subset \iota_{\gamma x \gamma^{-1}}^{-1}(\mathcal{O} \cap \iota_{\gamma x \gamma^{-1}}(\boldsymbol{k}))$. A similar argument gives the opposite inclusion and thus $\mathrm{cond}(x) = \mathrm{cond}(\iota_x) = \mathrm{cond}(\iota_{\gamma x \gamma^{-1}}) = \mathrm{cond}(\gamma x \gamma^{-1})$. □

This action is compatible with the action on $\mathcal{E}(c)$, therefore

$$\Gamma^* \backslash L(t,c) \xrightarrow{\sim} \Gamma^* \backslash \mathcal{E}(c). \tag{3.24}$$

To determine the set of Γ^*-orbits in $L(t,c)$, we examine the right-hand side of (3.24). Let R be the ring of integers of an imaginary quadratic field $\boldsymbol{k}$. Fix an embedding $\iota_0 : \boldsymbol{k} \hookrightarrow B$ with $\mathrm{cond}(\iota_0) = 1$, i.e. $\iota_0(R) \subset \mathcal{O}$. Since all embeddings of $\boldsymbol{k}$ into B are conjugate, there is a bijection

$$B^\times / \boldsymbol{k}^\times \xrightarrow{\sim} \mathcal{E},$$

$$\omega \mapsto Ad(\omega) \circ \iota_0.$$

Then

$$\Gamma^*\backslash B^\times/\boldsymbol{k}^\times \xrightarrow{\sim} \Gamma^*\backslash\mathcal{E}, \tag{3.25}$$

where the action of Γ^* on $B^\times/\boldsymbol{k}^\times$ is left multiplication. Define

$$B^\times(c) = \{\omega \in B^\times \mid \mathrm{cond}(Ad(\omega)\circ\iota_0) = c\} \tag{3.26}$$

so that

$$\Gamma^*\backslash B^\times(c)/\boldsymbol{k}^\times \xrightarrow{\sim} \Gamma^*\backslash\mathcal{E}(c). \tag{3.27}$$

Let Ord $=$ Ord(B) be the set of all maximal orders of B. For any $\mathcal{O} \in$ Ord, define the conductor of $\mathcal{O}$ to be the conductor of $\iota_0^{-1}(\mathcal{O}\cap\iota_0(\boldsymbol{k}))$. Define for $\omega \in B^\times$, $\mathcal{O}_\omega = \omega^{-1}\mathcal{O}\omega \in$ Ord.

Lemma 3.2.5. *The conductor of $\mathcal{O}_\omega$ is cond(ω).*

Proof. Consider the string of equalities:

$$\iota_0^{-1}(\mathcal{O}_\omega) = \iota_0^{-1}(\omega^{-1}\mathcal{O}\omega) = (Ad(\omega)\circ\iota_0)^{-1}(\mathcal{O}).$$

The order on the far left is the order that defines the conductor of $\mathcal{O}$ and the order on the far right defines the conductor of ω. Since the orders are the same, the conductors are equal. □

There is an action of $B^\times_{\mathbb{A}_f} = (B\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times$ on Ord via

$$\xi\cdot\mathcal{O}_\omega = \xi^{-1}\widehat{\mathcal{O}_\omega}\xi\cap B \tag{3.28}$$

where $\widehat{\mathcal{O}} = \mathcal{O}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$. In fact, $B^\times_{\mathbb{A}_f}$ acts transitively on Ord, and thus

$$\begin{aligned} \mathrm{N}_{B^\times_{\mathbb{A}_f}}(\widehat{\mathcal{O}})\backslash B^\times_{\mathbb{A}_f} &\xrightarrow{\sim} \mathrm{Ord}, \\ \xi &\mapsto \xi^{-1}\widehat{\mathcal{O}}\xi\cap B. \end{aligned}$$

Furthermore, the double cosets

$$\mathrm{N}_{B^{\times}_{\mathbb{A}_f}}(\widehat{\mathcal{O}})\backslash B^{\times}_{\mathbb{A}_f}/B^{\times} \tag{3.29}$$

correspond to the $B^{\times}$-conjugacy classes of the maximal orders in B.

Since B is an indefinite quaternion algebra, all maximal orders of B are conjugate. Thus

$$\mathrm{N}_{B^{\times}}(\mathcal{O})\backslash B^{\times} \xrightarrow{\sim} \mathrm{Ord}. \tag{3.30}$$

Let $\mathrm{Ord}(c) \subset \mathrm{Ord}$ be the subset of orders with conductor c. Then, with notations as before,

$$\mathrm{N}_{B^{\times}}(\mathcal{O})\backslash B^{\times}(c) \xrightarrow{\sim} \mathrm{Ord}(c). \tag{3.31}$$

Lemma 3.2.6. *If $\mathcal{O}_c \in Ord(c)$ and $\xi \in \boldsymbol{k}^{\times}_{\mathbb{A}_f} = (\boldsymbol{k}\otimes_{\mathbb{Q}}\mathbb{A}_f)^{\times}$, then $\xi^{-1}\widehat{\mathcal{O}_c}\xi\cap B \in Ord(c)$.*

Proof. Since $\mathcal{O}_c \in \mathrm{Ord}(c)$, then $\iota_0^{-1}(\mathcal{O}_c \cap \iota_0(\boldsymbol{k}))$ has conductor c. Consider $\mathcal{O}_\xi = \xi^{-1}\widehat{\mathcal{O}_c}\xi \cap \iota_0(\boldsymbol{k}) = \xi^{-1}(\widehat{\mathcal{O}_c} \cap \xi\iota_0(\boldsymbol{k})\xi^{-1})\xi$. Then, since $\xi \in \boldsymbol{k}^{\times}_{\mathbb{A}_f}$ it commutes with $\iota_0(\boldsymbol{k})$. So $\mathcal{O}_\xi = \xi^{-1}(\widehat{\mathcal{O}_c} \cap \iota_0(\boldsymbol{k}))\xi$. Now the conductor of $\xi^{-1}\widehat{\mathcal{O}_c}\xi \cap B$ is given by

$$\begin{aligned}
\iota_0^{-1}(\xi^{-1}\widehat{\mathcal{O}_c}\xi \cap B \cap \iota_0(\boldsymbol{k})) &= (Ad(\xi)\circ\iota_0)^{-1}(\widehat{\mathcal{O}_c} \cap B \cap \iota_0(\boldsymbol{k})) \\
&= (Ad(\xi)\circ\iota_0)^{-1}(\mathcal{O}_c \cap \iota_0(\boldsymbol{k})) \\
&= \iota_0^{-1}(\mathcal{O}_c \cap \iota_0(\boldsymbol{k})).
\end{aligned}$$

Hence the action of $\boldsymbol{k}^{\times}_{\mathbb{A}_f}$ preserves $\mathrm{Ord}(c)$. □

Theorem 3.2.7 (Chevalley-Hasse-Noether). *The group $\boldsymbol{k}^{\times}_{\mathbb{A}_f}$ acts transitively on $Ord(c)$ for any c.*

Corollary 3.2.8.

$$\Gamma^*\backslash L(t,c) \overset{\sim}{\leftrightarrow} N_{B^\times_{\mathbb{A}_f}}(\mathcal{O}_c) \cap \boldsymbol{k}^\times_{\mathbb{A}_f}\backslash \boldsymbol{k}^\times_{\mathbb{A}_f}/\boldsymbol{k}^\times. \tag{3.32}$$

Proof. From the theorem, for a given $\mathcal{O}_c \in \mathrm{Ord}(c)$ there is a bijection

$$\mathrm{N}_{B^\times_{\mathbb{A}_f}}(\mathcal{O}_c) \cap \boldsymbol{k}^\times_{\mathbb{A}_f}\backslash \boldsymbol{k}^\times_{\mathbb{A}_f} \overset{\sim}{\to} \mathrm{Ord}(c) \tag{3.33}$$

given by the orbit of $\mathcal{O}_c$. Then the composition of the bijections in (3.24), (3.27), and (3.31) yield the result. □

Let $\delta_0(\Delta, D)$ be the number of primes ramified in both B and $\boldsymbol{k}$ and define

$$\delta(\Delta, D) = \begin{cases} \delta_0(\Delta, D) - 1 & \text{if all ramified primes of } \boldsymbol{k} \text{ are ramified in } B \\ \delta_0(\Delta, D) & \text{otherwise} \end{cases}. \tag{3.34}$$

Theorem 3.2.9. *Let $R_c \in \boldsymbol{k}$ be the order of conductor c, then*

$$[\widehat{R_c}^\times\backslash \boldsymbol{k}^\times_{\mathbb{A}_f}/\boldsymbol{k}^\times : N_{B^\times_{\mathbb{A}_f}}(\mathcal{O}_c) \cap \boldsymbol{k}^\times_{\mathbb{A}_f}\backslash \boldsymbol{k}^\times_{\mathbb{A}_f}/\boldsymbol{k}^\times] = 2^{\delta(\Delta, D)}.$$

Proof. For a prime $p \nmid D$,

$$\mathrm{N}_{B^\times_p}(\mathcal{O}_c) = \mathcal{O}^\times_{c,p}\mathbb{Q}^\times_p$$

thus

$$\mathrm{N}_{B^\times_p}(\mathcal{O}_c) \cap \boldsymbol{k}^\times_p = R^\times_{c,p}\mathbb{Q}^\times_p. \tag{3.35}$$

For primes $p \mid D$, $\mathrm{N}_{B^\times_p}(\mathcal{O}_c) = B^\times_p$. When p is inert in $\boldsymbol{k}$, (3.35) still holds. However, when p in ramified in $\boldsymbol{k}$,

$$\mathrm{N}_{B^\times_p}(\mathcal{O}_c) \cap \boldsymbol{k}^\times_p = R^\times_{1,p}\mathbb{Q}^\times_p \cup R^\times_{1,p}\mathbb{Q}^\times_p\pi_p$$

where $\pi_p^2 = p$.

Altogether, then, there is a surjection

$$\widehat{R_c}^{\times} \backslash \boldsymbol{k}_{\mathbb{A}_f}^{\times} / \boldsymbol{k}^{\times} \twoheadrightarrow \mathrm{N}_{B_{\mathbb{A}_f}^{\times}}(\mathcal{O}_c) \cap \boldsymbol{k}_{\mathbb{A}_f}^{\times} \backslash \boldsymbol{k}_{\mathbb{A}_f}^{\times} / \boldsymbol{k}^{\times} \tag{3.36}$$

given by modding out by the subgroup generated by the elements $(1, ..., 1, \pi_p, 1, ...)$ for p ramified in both B and $\boldsymbol{k}$. The size of this subgroup is $2^{\delta(\Delta,D)}$. □

Corollary 3.2.10. *Let $h(c^2\Delta)$ be the ideal class number of the order of conductor c in the quadratic field of discriminant Δ and $-4t = n^2\Delta$ as before. Then*

$$|\Gamma^* \backslash L(t)| = 2^{-\delta(\Delta,D)} \sum_{\substack{c|n \\ (D,c)=1}} h(c^2\Delta). \tag{3.37}$$

Proof. This follows from recognizing $\widehat{R_c}^{\times} \backslash \boldsymbol{k}_{\mathbb{A}_f}^{\times} / \boldsymbol{k}^{\times}$ as the desired ideal class group.

□

Chapter 4

Borcherds Forms

4.1 The Space of Negative Planes of V

Recall that $V = \{x \in B \mid \mathrm{tr}(x) = 0\}$ has signature $(1, 2)$. Let $\mathfrak{D}$ be the space of oriented negative 2-planes in V. Call $[z_1, z_2] \in \mathfrak{D}$ a proper basis if $(z_1, z_1) = (z_2, z_2) = -1$ and $(z_1, z_2) = 0$. $\mathfrak{D}$ can be viewed in many other ways. Define

$$\mathcal{Q} = \{v \in V(\mathbb{C}) \mid (v, v) = 0, (v, \overline{v}) < 0\}/\mathbb{C}^\times. \tag{4.1}$$

This is an open subset of a quadric in $\mathbb{P}(V(\mathbb{C}))$. Also define $V_+(\mathbb{R})$ to be the set of vectors in $V(\mathbb{R})$ with positive norm. Recall that $B = \left(\frac{q,D}{\mathbb{Q}}\right)$ with $\alpha^2 = q$ and $\beta^2 = D$ and that there is a fixed embedding $\phi_D : B \hookrightarrow \mathrm{M}_2(\mathbb{R})$. Let V have the canonical basis $\{\alpha, \beta, \alpha\beta\}$.

Proposition 4.1.1. *There is a diagram of bijections as follows.*

$$\begin{array}{ccc} \mathfrak{h}^\pm & \xrightarrow{\phi} & V_+(\mathbb{R})/\mathbb{R}_{>0} \\ w\downarrow & & \downarrow\rho \\ \mathcal{Q} & \xleftarrow{\sigma} & \mathfrak{D}(\mathbb{R}) \end{array} \tag{4.2}$$

where the maps are given by

$$\phi(u+iv) = -\left(\frac{\sqrt{D}(q-u^2-v^2)}{2q}\right)\alpha + u\beta + \left(\frac{u^2+v^2+q}{2q}\right)\alpha\beta, \tag{4.3}$$

$$w(z) = \left(\frac{q-z^2}{2q}\right)\alpha + \left(\frac{z}{\sqrt{D}}\right)\beta + \left(\frac{q+z^2}{2q\sqrt{D}}\right)\alpha\beta, \tag{4.4}$$

$$\sigma([z_1, z_2]) = z_1 - iz_2, \tag{4.5}$$

$$\rho(x) = x^{\perp}. \tag{4.6}$$

Proof. It will suffice to show that $(\sigma \circ \rho \circ \phi)(u+iv) = w(u+iv)$. This first involves finding the plane perpendicular to $\phi(u+iv)$. This plane is given, in a proper basis, by

$$z_1 = -\left(\frac{u}{q}\right)\alpha + \left(\frac{1}{\sqrt{D}}\right)\beta + \left(\frac{u}{q\sqrt{D}}\right)\alpha\beta, \tag{4.7}$$

$$z_2 = \left(\frac{q-u^2+v^2}{2qv}\right)\alpha + \left(\frac{u}{v\sqrt{D}}\right)\beta + \left(\frac{q+u^2-v^2}{2qv\sqrt{D}}\right)\alpha\beta. \tag{4.8}$$

Then one can see that $z_1 - iz_2 = w(u+iv)$ after scaling by $iv \in \mathbb{C}^{\times}$. □

4.2 Rational Quadratic Divisors

The previous section shows that $\mathfrak{h}^{\pm} \simeq \mathfrak{D}(\mathbb{R})$. Write $\mathfrak{D} = \mathfrak{D}^+ \cup \mathfrak{D}^-$ where $\mathfrak{D}^+$ (resp., $\mathfrak{D}^-$) are the planes with positive (negative) orientation. For $x \in V(\mathbb{Q})$ define

$$\mathfrak{D}_x = \{z \in \mathfrak{h}^{\pm} \mid (x, w(z)) = 0\}. \tag{4.9}$$

By (4.4), for $x = x_1\alpha + x_2\beta + x_3\alpha\beta$,

$$(x, w(z)) = \left(\frac{x_1 + x_3\sqrt{D}}{2}\right)z^2 - (x_2\sqrt{D})z - \frac{q(x_1 - x_3\sqrt{D})}{2}. \tag{4.10}$$

Hence

$$\mathfrak{D}_x = \left\{\frac{x_2\sqrt{D} \pm \sqrt{-Q(x)}}{x_1 + x_3\sqrt{D}}\right\}. \tag{4.11}$$

Let $\mathfrak{D}_x^{\pm} = \mathfrak{D}_x \cap \mathfrak{D}^{\pm}$.

Recall that $\mathrm{PGL}_2(\mathbb{R})$ acts on $\mathfrak{h}^{\pm}$ via fractional linear transformations, and a $B^{\times}$ action can be defined via the embedding $\phi_D : B \hookrightarrow \mathrm{M}_2(\mathbb{R})$.

Proposition 4.2.1. *For $x \in V$ with $Q(x) > 0$, $\mathfrak{D}_x$ is the set of fixed points of x.*

Proof. Let $x = x_1\alpha + x_2\beta + x_3\alpha\beta$. Then

$$\phi_D(x) = \begin{pmatrix} x_2\sqrt{D} & q(x_1 - x_3\sqrt{D}) \\ x_1 + x_3\sqrt{D} & -x_2\sqrt{D} \end{pmatrix}. \tag{4.12}$$

A fixed point, z, of this matrix satisfies

$$zx_2\sqrt{D} + q(x_1 - x_3\sqrt{D}) = z^2(x_1 + x_3\sqrt{D}) - zx_2\sqrt{D}. \tag{4.13}$$

This is equivalent to (4.10). □

Definition 4.2.1. *Let $G = \Gamma$ or Γ^* and let $G\eta$ denote the G-orbit of $\eta \in L^\vee/L$. The rational quadratic divisor $Z(d, \eta; G)$ is given by*

$$Z(d, \eta; G) = \sum_{\substack{x \in L^\vee \cap V(d) \\ x + L \in G\eta \\ \bmod G}} \mathrm{pr}_G(\mathfrak{D}_x^+), \tag{4.14}$$

where $\mathrm{pr}_G : \mathfrak{D}^+ \to G\backslash\mathfrak{D}^+$.

For more details on this definition in the case of $G = \Gamma$, see the Appendix of [9].

4.3 Borcherds Forms

Let $H = \mathrm{GSpin}(V)$. Viewed as an algebraic group, $H(\mathcal{A}) \simeq (B \otimes_{\mathbb{Q}} \mathcal{A})^\times$ for any $\mathbb{Q}$-algebra $\mathcal{A}$. Let $K \subset H(\mathbb{A}_f)$ be a compact open set such that $H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K$ where $H(\mathbb{R})^+$ is the component of $H(\mathbb{R})$ that contains the identity.

Definition 4.3.1. *A modular form of weight $k \in \mathbb{Z}$ on $\mathfrak{D} \times H(\mathbb{A}_f)/K$ is a function $\Psi : \mathfrak{D} \times H(\mathbb{A}_f)/K \to \mathbb{C}$ such that*

$$\Psi(\gamma z, \gamma h) = j(\gamma, z)^k \Psi(z, h)$$

for all $\gamma \in H(\mathbb{Q})$, where $j(\gamma, z)$ is the automorphy factor given on page 11 of [8].

Note that in the cases we will focus on, $k = 0$ and the automorphy factor will be inconsequential.

Let L be a lattice and F be a modular form valued in $\mathbb{C}[L^\vee/L]$ with Fourier expansion given by

$$F(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\eta(m) \mathbf{q}^m e_\eta \tag{4.15}$$

where $\{e_\eta\}_{\eta \in L^\vee/L}$ form the basis of $\mathbb{C}[L^\vee/L]$. Since Γ and Γ^* act on $L^\vee/L$, they also act via linearity on the algebra $\mathbb{C}[L^\vee/L]$ and F.

Definition 4.3.2. *For a lattice L with signature $(n, 2)$, a Borcherds form $\Psi(F)$ is a meromorphic modular form on $\mathfrak{D} \times H(\mathbb{A}_f)/K$ arising from the regularized theta lift of a weight $1 - \frac{n}{2}$ meromorphic modular form F as in (4.15) with $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$. See [11], [8], [2].*

Borcherds forms have the following key properties.

Theorem 4.3.1 (Theorem 1.3 of [8]). *Assume F is given as in (4.15) and is Γ^* invariant.*

1) The weight of $\Psi(F)$ is $c_0(0)$.

2) $\operatorname{div}(\Psi(F)^2) = \sum_{\eta \in L^\vee/L} \sum_{m > 0} c_\eta(-m) Z(m, \eta; \Gamma^*)$.

4.4 Adelic View

We can rephrase some of the definitions from Section 2.3 from an adelic point of view. This will allow the machinary of Borcherds forms to apply to the computation of singular moduli on $\mathcal{X}_D$ and $\mathcal{X}_D^*$.

Let K_Γ be the compact open set $\widehat{\mathcal{O}}^\times \subset H(\mathbb{A}_f)$. Then

$$\Gamma = H(\mathbb{Q}) \cap H(\mathbb{R})^+ K_\Gamma. \tag{4.16}$$

Let K_{Γ^*} be defined analagously. Then $\mathcal{X}_D$ and $\mathcal{X}_D^*$ are given by

$$\mathcal{X}_D \simeq \Gamma\backslash\mathfrak{D} \simeq H(\mathbb{Q})\backslash(\mathfrak{D} \times H(\mathbb{A}_f)/K_\Gamma), \tag{4.17}$$

$$\mathcal{X}_D^* \simeq \Gamma^*\backslash\mathfrak{D} \simeq H(\mathbb{Q})\backslash(\mathfrak{D} \times H(\mathbb{A}_f)/K_{\Gamma^*}). \tag{4.18}$$

Notice that $\mathcal{X}_D$ and $\mathcal{X}_D^*$ are natural domains for weight-0 Borcherds forms.

The CM points can be viewed adelically as well. An element $x \in V_+(\mathbb{Q})$ gives rise to the decomposition of V as $V = \mathbb{Q}x \oplus U$ where $U = x^\perp$ is a negative plane. This splitting corresponds to a two-point set $\mathfrak{D}_x$. As a rational inner product space, $U \simeq \boldsymbol{k}$ for some quadratic imaginary field $\boldsymbol{k}$ with quadratic form given by a constant times the norm on $\boldsymbol{k}$. Set $T \simeq \mathrm{GSpin}(U)$. Then, with ι_x as in Section 3.2, $T(\mathbb{Q}) \simeq \iota_x(\boldsymbol{k}^\times) \subset H(\mathbb{Q})$ and the CM points are the image of

$$Z_{\Gamma^*}(U) = T(\mathbb{Q})\backslash(\mathfrak{D}_x \times T(\mathbb{A}_f)/K_{\Gamma^*}) \hookrightarrow \mathcal{X}_D^*. \tag{4.19}$$

In addition, there is an identification

$$Z_{\Gamma^*}(U) \overset{\sim}{\leftrightarrow} \Gamma^*\backslash L(t,c) \tag{4.20}$$

where $\boldsymbol{k} \simeq \mathbb{Q}(\sqrt{-t})$ as in Section 3.2. Thus

$$|Z_{\Gamma^*}(U)| = \frac{h(\boldsymbol{k})}{2^{\delta(\Delta,D)}}. \tag{4.21}$$

4.5 Borcherds Forms at CM Points

Let Recall that $L = \mathcal{O} \cap V$ is a lattice in V corresponding to a fixed maximal order $\mathcal{O}$. Then there are sublattices

$$L_+ = \mathbb{Q}x \cap L, \tag{4.22}$$

$$L_- = U \cap L. \tag{4.23}$$

In general, $L \neq L_- + L_+$, and

$$L_- + L_+ \subseteq L \subseteq L^\vee \subseteq L_-^\vee + L_+^\vee.$$

Hence an element $\eta \in L^\vee$ decomposes as $\eta = \eta_- + \eta_+$ for $\eta_\pm \in L_\pm^\vee$.

Since $U \simeq \boldsymbol{k}$ there is an associated quadratic character χ_Δ given by the Legendre symbol, $\chi_\Delta(n) = \left(\frac{\Delta}{n}\right)$, where Δ is the discriminant of $\boldsymbol{k}$.

Definition 4.5.1 ([11]). *For $\mu \in L_-^\vee / L_-$ and $\psi_\mu = char(\mu + L_-)$, let $E(\tau, s; \psi_\mu, +1)$ be the incoherent Eisenstein series of weight 1 with Fourier expansion*

$$E(\tau, s; \psi_\mu, +1) = \sum_m A_\mu(s, m, v)\mathbf{q}^m \tag{4.24}$$

where the Fourier coefficients have Laurent expansions

$$A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2) \tag{4.25}$$

at $s = 0$. Then for $\eta \in L^\vee / L$ and $m \in \mathbb{Q}$ define,

$$\kappa_\eta(m) = \sum_{\lambda \in L/(L_+ + L_-)} \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa^-_{\eta_- + \lambda_-}(m - Q(x)) \tag{4.26}$$

where

$$
\begin{aligned}
\kappa_\mu^-(m') &= \begin{cases} \lim_{v\to\infty} b_\mu(m',v) & \text{if } m'>0 \\ k_0(0)\psi_\mu(0) & \text{if } m'=0 \\ 0 & \text{if } m'<0 \end{cases}, && (4.27)\\
k_0(0) &= \log(|\Delta|) + 2\frac{\Lambda'(1+\chi_\Delta)}{\Lambda(1,\chi_\Delta)}, && (4.28)
\end{aligned}
$$

and $\Lambda(s,\chi_\Delta)$ *is the normalized L-series* $\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_\Delta)$.

Theorem 4.5.1 (Corollary 3.4 of [11]). *Assume* $c_\eta(m)\in\mathbb{Z}$ *for* $m\le 0$, $c_0(0)=0$, *and that the* 0*-cycle* $Z_{\Gamma^*}(U)$ *defined in (4.19) does not meet the divisor of* $\Psi(F)$. *Then*

$$\frac{1}{|Z_{\Gamma^*}(U)|}\sum_{z\in Z_{\Gamma^*}(U)} \log||\Psi(z,f)||^2 = \frac{-1}{2^{d(B)}}\sum_\eta\sum_{m\ge 0} c_\eta(m)\kappa_\eta(m) \qquad (4.29)$$

where $h(\boldsymbol{k})$ *is the ideal class number of the quadratic field* $\boldsymbol{k}\simeq U$.

This important theorem will be used to compute the norms of singular moduli in Chapter 7. However, first a supply of appropriate vector-valued modular forms F are needed to serve as the input to the Borcherds construction of $\Psi(F)$.

Chapter 5

Input Forms

This chapter is presented in general terms. However, rather than appearing redundant, the notation implies how the general theory applies to the set-up in Chapters 2 through 4.

5.1 $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ and the Weil Representation

The Lie group $\mathrm{SL}_2(\mathbb{R})$ has a double cover $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ with elements of the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d}\right).$$

The group structure is given by

$$(G_1, j_1(\cdot))(G_2, j_2(\cdot)) = (G_1G_2, j_1(G_2(\cdot))j_2(\cdot)).$$

The group $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ is defined as the inverse image in $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{Z})$ and is generated by the two elements

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right), \tag{5.1}$$

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right), \tag{5.2}$$

which satisfy

$$Z = S^2 = (ST)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right). \tag{5.3}$$

The element Z generates the center of $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ and the quotient by Z^2 is $\mathrm{SL}_2(\mathbb{Z})$. Also, $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ acts on $\mathfrak{h}^\pm$ via its image in $\mathrm{SL}_2(\mathbb{Z})$. Throughout the following, let

$$\gamma = \gamma^\pm = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) \in \widetilde{\mathrm{SL}_2}(\mathbb{Z}). \tag{5.4}$$

Let L be a lattice with quadratic form Q' and let $L^\vee$ be the dual lattice under the associated inner product as in (3.2). To ease notation, let $\Lambda_L = L^\vee/L$. Then Milgram's formula gives $\mathrm{sign}(L)$, the signature mod 8 of L, via

$$\sum_{\eta \in \Lambda_L} \mathbf{e}(Q'(\eta)) = \sqrt{|\Lambda_L|}\mathbf{e}(\mathrm{sign}(L)/8) \tag{5.5}$$

where $\mathbf{e}(x) = e^{2\pi i x}$. For $\eta \in \Lambda_L$, let e_η denote the corresponding basis element in the group ring $\mathbb{C}[\Lambda_L]$. In [2], Borcherds defines the Weil representation ρ_{Λ_L} on the generators of $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ by

$$\overline{\rho}_{\Lambda_L}(T)e_\eta = \mathbf{e}(Q'(\eta))e_\eta, \tag{5.6}$$

$$\overline{\rho}_{\Lambda_L}(S)e_\eta = \frac{\mathbf{e}(-\mathrm{sign}(L)/8)}{\sqrt{|\Lambda_L|}} \sum_{\delta \in \Lambda_L} \mathbf{e}(-(\eta, \delta))e_\delta. \tag{5.7}$$

However, we will use the dual representation $\rho_{\Lambda_L} = \overline{\rho}_{\Lambda_L}^\vee$ since the quadratic form in Chapters 2 through 4 is actually given by $Q(x) = -Q'(x)$. On the generators ρ_{Λ_L} is given by

$$\rho_{\Lambda_L}(T)e_\eta = \mathbf{e}(-Q(\eta))e_\eta, \tag{5.8}$$

$$\rho_{\Lambda_L}(S)e_\eta = C_L \sum_{\delta \in \Lambda_L} \mathbf{e}(-(\eta, \delta))e_\delta \tag{5.9}$$

where

$$C_L = \frac{\mathbf{e}(\operatorname{sign}(L)/8)}{\sqrt{|\Lambda_L|}} = \frac{\sum_{\eta\in\Lambda_L} \mathbf{e}(Q(\eta))}{|\Lambda_L|}. \tag{5.10}$$

(This approach follows [11] and [8]. However most of the results in this section are the dualized versions of those found in [3].) Define the level of Λ_L to be the smallest integer N such that $NQ(\eta) \in \mathbb{Z}$ for all $\eta \in L^\vee$. Then the representation ρ_{Λ_L} factors through $\widetilde{\mathrm{SL}_2}(\mathbb{Z}/N\mathbb{Z})$, the double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

The Kronecker symbol $\left(\frac{c}{d}\right)$ for coprime c and d is multiplicative in both integral entries and is an extension of the Legendre symbol with

$$\left(\frac{c}{-1}\right) = \begin{cases} -1 & c < 0 \\ 1 & c > 0 \end{cases}, \tag{5.11}$$

$$\left(\frac{c}{2}\right) = \begin{cases} 0 & c \text{ even} \\ 1 & c \equiv \pm 1 \mod 8 \\ -1 & c \equiv \pm 3 \mod 8 \end{cases}, \tag{5.12}$$

$$\left(\frac{0}{\pm 1}\right) = \left(\frac{\pm 1}{0}\right) = 1. \tag{5.13}$$

Define the congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ as the preimage of the upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\widetilde{\Gamma_0}(N)$ as its inverse image in $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$.

Definition 5.1.1 ([3]). *For* $\gamma \in \widetilde{\Gamma_0}(N)$ *define*

$$\chi_n(\gamma) = \left(\frac{d}{n}\right), \tag{5.14}$$

$$\chi_\theta(\gamma^\pm) = \begin{cases} \pm\left(\frac{c}{d}\right) & d \equiv 1 \mod 4 \\ \mp i \left(\frac{c}{d}\right) & d \equiv 3 \mod 4 \end{cases}, \tag{5.15}$$

$$\chi_L(\gamma) = \begin{cases} \left(\chi_\theta^{-sign(L)+\left(\frac{-1}{|\Lambda_L|}\right)-1} \chi_{|\Lambda_L| 2^{sign(L)}}\right)(\gamma) & 4 \mid N \\ \chi_{|\Lambda_L|}(\gamma) & 4 \nmid N \end{cases} . \tag{5.16}$$

Theorem 5.1.1. *Suppose Λ_L has level N. If b and c are divisible by N then $\gamma \in \widetilde{SL_2}(\mathbb{Z})$ acts on $\mathbb{C}[\Lambda_L]$ by*

$$\rho_{\Lambda_L}(\gamma) e_\eta = \chi_L(\gamma) e_{a\eta}.$$

Proof. This is essentially the same theorem as Theorem 5.4 of [3]. However, since we are using the dual representation,

$$Z(e_\eta) = (-i)^{-\text{sign}(L)} e_\eta, \tag{5.17}$$

which gives χ_L as defined in (5.16). □

Corollary 5.1.2. *Suppose Λ_L has level N and that $\eta \in \Lambda_L$ has norm 0. Then $\gamma \in \widetilde{\Gamma_0}(N)$ acts on the element e_η by*

$$\rho_{\Lambda_L}(\gamma) e_\eta = \chi_L(\gamma) e_{a\eta}.$$

Proof. Any element $\gamma \in \widetilde{\Gamma_0}(N)$ can be written as

$$\gamma = T^n \left(\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right)$$

where N divides c and b'. Then χ_L is trivial on T. Since $a' \equiv a \mod N$ and the order of η divides N, $a'\eta = a\eta$. □

5.2 The Cusps of $\Gamma_0(N)$

In this section we examine the coset representatives of

$$\widetilde{\Gamma_0}(N) \backslash \widetilde{\mathrm{SL}_2}(\mathbb{Z}) \simeq \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})$$

through the relationship between coset representatives of $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$ and the cusps of $\Gamma_0(N)$.

Proposition 5.2.1. *Given a complete set* $\{a_i/c_i\}_{i=0}^{I}$ *of inequivalent cusps for* $\Gamma_0(N)$, *choose integers* b_i, d_i *such that* $a_i d_i - b_i c_i = 1$. *Then a complete set of coset representatives for* $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ *is given by*

$$\left\{ \begin{pmatrix} a_i & a_i n + b_i \\ c_i & c_i n + d_i \end{pmatrix} \middle| \, 0 \le n < N/\gcd(N, c_i^2), 0 \le i \le I \right\}. \tag{5.18}$$

Proof. Since both sets are left invariant under $\Gamma_0(N)$ there is no need to show well-definition; it suffices to show that the set of coset representatives is complete. A coset representative $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfies $\gcd(c', d') = 1$ and two such representatives are equivalent if and only if $c_1' d_2' \equiv c_2' d_1' \mod N$. Thus

$$\begin{pmatrix} a_i & a_i n + b_i \\ c_i & c_i n + d_i \end{pmatrix} \sim \begin{pmatrix} a_i & a_i n' + b_i \\ c_i & c_i n' + d_i \end{pmatrix} \quad \Leftrightarrow \quad n \equiv n' \mod N/\gcd(N, c_i^2). \tag{5.19}$$

Hence (5.18) is a complete set of coset representatives. □

Lemma 5.2.2 (Lemma 2.2.3 of [4]). *For* $i \in \{1, 2\}$, *let* $\alpha_i = p_i/q_i$ *be two cusps written in lowest terms. The following are equivalent.*

1) $\alpha_2 = \gamma(\alpha_1)$ *for some* $\gamma \in \Gamma_0(N)$.

2) $q_2 \equiv u q_1 \mod N$ *and* $u p_2 \equiv p_1 \mod \gcd(q_1, N)$ *with* $\gcd(u, N) = 1$.

3) $s_1 q_2 \equiv s_2 q_1 \mod \gcd(q_1 q_2, N)$ *where* s_j *satisfies* $p_j s_j \equiv 1 \mod q_j$.

Theorem 5.2.3. *Every cusp of* $\Gamma_0(N)$ *is equivalent to a cusp of the form* $1/c$ *for some* $c \in \mathbb{Z}$.

Proof. Suppose p/q is a cusp of $\Gamma_0(N)$ in lowest terms. It suffices to find $u \in \mathbb{Z}$ such that $u \equiv p \mod \gcd(q, N)$ and $\gcd(u, N) = 1$, because then, by the second part of Lemma 5.2.2, $p/q \sim -1/uq$. Finding such a u is equivalent to finding a $k \in \mathbb{Z}$ such that $\gcd(p + k\gcd(q, N), N) = 1$. Let l be a prime dividing N. If $l \mid \gcd(q, N)$, then $l \nmid p$ since $\gcd(p, q) = 1$. Thus $l \nmid p + k\gcd(q, N)$ for every $k \in \mathbb{Z}$. Now suppose $l \nmid \gcd(q, N)$. Let k_l be given by $k_l \equiv 1 - \frac{p}{\gcd(q,N)} \mod l$. Then, by Chinese Remainder Theorem, there exists $k \in \mathbb{Z}$ with $k \equiv k_l \mod l$ for every $l \mid N$, $l \nmid \gcd(q, N)$. Then this k satisfies $\gcd(p + k\gcd(q, N), N) = 1$ and $p/q \sim 1/(pq + kq\gcd(q, N))$. □

Corollary 5.2.4. *The coset representatives of* $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ *can all be chosen of the form*

$$\gamma(m,n) = S^{-1}T^{-n}S^{-1}T^{-m} = \begin{pmatrix} -1 & 0 \\ -n & -1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \tag{5.20}$$

$$= \begin{pmatrix} -1 & m \\ -n & -1 + mn. \end{pmatrix} \tag{5.21}$$

with $0 \leq m < N/\gcd(N, n^2)$.

5.3 Vector-Valued Modular Forms

Define the slash operator of weight k for an element $\gamma \in \widetilde{\mathrm{SL}_2}(\mathbb{Z})$ by

$$f|^k_{\gamma^\pm}(\tau) = (\pm\sqrt{c\tau + d})^{2k} f(\gamma\tau).$$

The slash operator satisfies

$$f|^k_{(\gamma_1\gamma_2)} = \left(f|^k_{\gamma_1}\right)|^k_{\gamma_2}.$$

Definition 5.3.1. *Suppose ρ is a representation of $\Gamma \subset \widetilde{SL_2}(\mathbb{Z})$ on a finite dimensional complex vector space $\mathcal{V}$. Then $F : \mathfrak{h}^{\pm} \to \mathcal{V}$ is a vector-valued modular form on Γ of weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ if it is meromorphic and satsifies*

$$F(\gamma^{\pm}\tau) = (\pm\sqrt{c\tau + d})^{2k}\rho(\gamma^{\pm})F(\tau) \tag{5.22}$$

for all $\gamma \in \Gamma$.

Definition 5.3.2. *Suppose f is a scalar-valued weight k modular form on $\widetilde{\Gamma_0}(N)$ with character χ_L. Then define a weight k modular form $F_f(\tau)$ valued in $\mathbb{C}[L^\vee/L]$ via*

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma_0}(N)\backslash \widetilde{SL_2}(\mathbb{Z})} f|^k_\gamma(\tau)\rho_{\Lambda_L}(\gamma^{-1})e_0. \tag{5.23}$$

Proposition 5.3.1. *$F_f(\tau)$ is well-defined.*

Proof. Suppose $\gamma_0\gamma$ for some $\gamma_0 \in \widetilde{\Gamma_0}(N)$ was chosen instead of the coset representative γ. Then the summand in (5.23) becomes

$$\begin{aligned} f|^k_{\gamma_0\gamma}(\tau)\rho_{\Lambda_L}((\gamma_0\gamma)^{-1})e_0 &= \chi_L(\gamma_0)f|^k_\gamma(\tau)\rho_{\Lambda_L}(\gamma^{-1})\rho_{\Lambda_L}(\gamma_0^{-1})e_0 \\ &= \chi_L(\gamma_0)f|^k_\gamma(\tau)\rho_{\Lambda_L}(\gamma^{-1})\chi_L(\gamma_0)^{-1}e_0 \\ &= f|^k_\gamma(\tau)\rho_{\Lambda_L}(\gamma^{-1})e_0. \end{aligned}$$

Thus $F_f(\tau)$ is independent of the choice of coset representatives. □

Proposition 5.3.2. *$F_f(\tau)$ is a modular form of type ρ_{Λ_L} and weight k on $\widetilde{SL_2}(\mathbb{Z})$.*

Proof. As usual, let $\gamma_1^{\pm} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) \in \widetilde{\mathrm{SL}_2}(\mathbb{Z})$. Then

$$\begin{aligned}
F_f(\gamma_1^{\pm}\tau) &= \sum_{\gamma} f|_{\gamma}^{k}(\gamma_1^{\pm}\tau)\rho_{\Lambda_L}(\gamma^{-1})e_0 && (5.24)\\
&= \sum_{\gamma} f|_{(\gamma_1^{\pm})^{-1}\gamma}^{k}(\gamma_1^{\pm}\tau)\rho_{\Lambda_L}(\gamma_1^{\pm}\gamma^{-1})e_0 && (5.25)\\
&= (\pm\sqrt{c\tau + d})^{2k}\rho_{\Lambda_L}(\gamma_1^{\pm}) \sum_{\gamma} f|_{\gamma}^{k}(\tau)\rho_{\Lambda_L}(\gamma^{-1})e_0 && (5.26)\\
&= (\pm\sqrt{c\tau + d})^{2k}\rho_{\Lambda_L}(\gamma_1^{\pm})F_f(\tau). && (5.27)
\end{aligned}$$

Hence (5.22) is satisfied. □

Proposition 5.3.3. *Let F_f have Fourier expansion as in (4.15). If $m + Q(\eta) \notin \mathbb{Z}$, then $c_\eta(m) = 0$.*

Proof. Since F_f is a modular form,

$$\begin{aligned}
F_f(\tau + 1) &= \rho_{\Lambda_L}(T)F_f(\tau) && (5.28)\\
\sum_{\eta\in\Lambda_L} \sum_{m\in\mathbb{Q}} c_\eta(m)\mathbf{q}^m\mathbf{e}(m)e_\eta &= \sum_{\eta\in\Lambda_L} \sum_{m\in\mathbb{Q}} c_\eta(m)\mathbf{q}^m\rho_{\Lambda_L}(T)e_\eta && (5.29)\\
&= \sum_{\eta\in\Lambda_L} \sum_{m\in\mathbb{Q}} c_\eta(m)\mathbf{q}^m\mathbf{e}(-Q(\eta))e_\eta. && (5.30)
\end{aligned}$$

Thus $m + Q(\eta) \notin \mathbb{Z}$ implies $c_\eta(m) = 0$. □

Proposition 5.3.4. *If f has no poles at finite cusps, then, for F_f as in (4.15), $c_\eta(m) = 0$ for $m < 0$ and $\eta \neq 0$.*

Proof. If f does not have a pole at a finite cusp, then the coordinate function $f|_{\gamma}^{k}$ in (5.23) can have a pole only when $\gamma(\infty) = \infty$. However, this is satisfied only by the coset representatives $\gamma(0, m)$. Since $0 \leq m < N/\gcd(N, 0) = 1$, $f|_{\gamma}^{k}$ can have a pole only if γ is the trivial coset representative, which has $\rho_{\Lambda_L}(\gamma^{-1})e_0 = e_0$. □

Now define $\Lambda_{L,n}$ to be the set of n-torsion points and define Λ_L^n via the exact sequence

$$0 \to \Lambda_{L,n} \to \Lambda_n \to \Lambda_L^n \to 0,$$

and

$$\Lambda_L^{n*} = \{\delta \in \Lambda_L^n \mid \ (\delta, \eta) = -nQ(\eta) \quad \forall \eta \in \Lambda_{L,n} \ \}.$$

Lemma 5.3.5. *For a fixed n, either $\Lambda_L^{n*} = \varnothing$ or the membership of δ into Λ_L^{n*} is completely determined by $Q(\delta)$.*

Proof. It suffices to examine the criteria locally at the primes that divide the level N. Recall from Section 3.1 that for an odd prime p, $\Lambda_{L,p} \simeq \mathbb{F}_{p^2}$ and $Q : \Lambda_{L,p} \to (1/p)\mathbb{Z}/\mathbb{Z}$. If $p \mid n$, then $(\Lambda_{L,p})_n = \Lambda_{L,p}$ and $(\Lambda_{L,p})^n = \{0\}$. Since $nQ(\delta) = 0 = (0, \delta)$ for all $\delta \in (\Lambda_{L,p})_n$, then $(\Lambda_{L,p})^{n*} = \{0\}$. If $p \nmid n$, then $(\Lambda_{L,p})_n = \{0\}$ and $(\Lambda_{L,p})^n = \Lambda_{L,p}$. Since $nQ(0) = 0 = (\delta, 0)$ for all $\delta \in (\Lambda_{L,p})^n = \Lambda_{L,p}$, then $(\Lambda_{L,p})^{n*} = \Lambda_{L,p}$. So for odd $p \mid N$,

$$(\Lambda_{L,p})^{n*} = \begin{cases} \{\delta \mid Q(\delta) \in (1/p)\mathbb{Z}_p/\mathbb{Z}_p\} & p \nmid n \\ \{0\} & p \mid n \end{cases}. \tag{5.31}$$

Now consider $p = 2$ where $\Lambda_{L,2} \simeq \mathbb{F}_4 \oplus \mathbb{F}_2$ and $Q : \Lambda_{L,2} \to (1/4)\mathbb{Z}/\mathbb{Z}$. Suppose $2 \nmid n$. Then $(\Lambda_{L,2})_n = \{0\}$, and $(\Lambda_{L,2})^n = \Lambda_{L,2}$. Since $nQ(0) = 0 = (\delta, 0)$ for all $\delta \in (\Lambda_{L,2})^n = \Lambda_{L,2}$, then $(\Lambda_{L,2})^{n*} = \Lambda_{L,2}$. Now suppose $2 \mid n$. Then $(\Lambda_{L,2})_n = \Lambda_{L,2}$, and $(\Lambda_{L,2})^n = \{0\}$. However, $nQ(\delta) = 0 = (0, \delta)$ for all $\delta \in (\Lambda_{L,p})$ only when $4 \mid n$.

Thus

$$(\Lambda_{L,2})^{n*} = \begin{cases} \{\delta \mid Q(\delta) \in (1/4)\mathbb{Z}_2/\mathbb{Z}_2\} & 2 \nmid n \\ \varnothing & 2 \parallel n \\ \{0\} & 4 \mid n \end{cases} . \tag{5.32}$$

Combining (5.31) and (5.32) into one global statement yields

$$\Lambda_L^{n*} = \begin{cases} \{\delta \mid Q(\delta) \in \left(\frac{\gcd(n,N)}{N}\right)\mathbb{Z}/\mathbb{Z}\} & 2 \not\parallel n \\ \varnothing & 2 \parallel n \end{cases} . \tag{5.33}$$

Thus the membership of an element is determined by its image under Q. □

Lemma 5.3.6 (Lemma 3.1 of [3]). *The sum*

$$\mathcal{S}_n(\delta) = \sum_{\eta \in \Lambda_L} \mathbf{e}(-(\eta, \delta) - nQ(\eta))$$

is equal to 0 *when* $\delta \notin \Lambda_L^{n*}$ *and has magnitude* $\sqrt{|\Lambda_L||\Lambda_{L,n}|}$ *otherwise.*

Proof. Computing the square of the absolute value of the sum gives

$$\begin{aligned} |\mathcal{S}_n(\delta)|^2 &= \sum_{\eta_1,\eta_2 \in \Lambda_L} \mathbf{e}(-(\eta_1, \delta) - nQ(\eta_1) + (\eta_2, \delta) + nQ(\eta_2)) & (5.34) \\ &= \sum_{\eta_1,\eta_2 \in \Lambda_L} \mathbf{e}((\eta_1, \delta) + nQ(\eta_1) + n(\eta_1, \eta_2)) & (5.35) \\ &= |\Lambda_L| \sum_{\eta_1 \in \Lambda_{L,n}} \mathbf{e}((\eta_1, \delta) + nQ(\eta_1)). & (5.36) \end{aligned}$$

Then, since a character integrated against the group is trivial (resp., the size of the group) if the character is nontrivial (trivial), this yields the result. □

Lemma 5.3.7 (Lemma 3.2 of [3]). *For* $\gamma \in \widetilde{SL_2}(\mathbb{Z})$ *as in (5.4),* $\rho_{\Lambda_L}(\gamma)e_0$ *is a linear combination of the elements* e_δ *for* $\delta \in \Lambda_L^{c*}$.

Proof. By Corollary 5.2.4, it is sufficient to prove this for γ of the form T^mST^nS for some $m, n \in \mathbb{Z}$ with $(N, n) = (N, c)$ since any γ is a product of an element of this form with an element of $\widetilde{\Gamma_0}(N)$ on the right, but e_0 is an eigenvector for $\widetilde{\Gamma_0}(N)$. Then

$$\begin{aligned}
\rho_{\Lambda_L}(S)e_0 &= C_L \sum_{\delta\in\Lambda_L} e_\delta, &(5.37)\\
\rho_{\Lambda_L}(T^nS)e_0 &= C_L \sum_{\delta\in\Lambda_L} \mathbf{e}(-nQ(\delta))e_\delta, &(5.38)\\
\rho_{\Lambda_L}(ST^nS)e_0 &= C_L^2 \sum_{\delta\in\Lambda_L}\sum_{\delta'\in\Lambda_L} \mathbf{e}(-nQ(\delta)-(\delta',\delta))e_{\delta'} &(5.39)\\
&= C_L^2 \sum_{\delta\in\Lambda_L^{n*}} \mathcal{S}_n(\delta)e_\delta, &(5.40)\\
\rho_{\Lambda_L}(T^mST^nS)e_0 &= C_L^2 \sum_{\delta\in\Lambda_L^{n*}} \mathcal{S}_n(\delta)\mathbf{e}(-mQ(\delta))e_\delta. &(5.41)
\end{aligned}$$

□

Theorem 5.3.8. *If $Q(\delta) = Q(\delta')$, then the e_δ and $e_{\delta'}$ components of F_f are equal.*

Proof. This follows from the fact that the coefficient $\mathcal{S}_n(\delta)\mathbf{e}(-mQ(\delta))$ depends only on $Q(\delta)$ which, by Proposition 5.3.5, is the same for all $\delta \in \Lambda_L^{n*}$. □

Corollary 5.3.9. *The modular form F_f is Γ^* invariant.*

Proof. This follows from the theorem and Proposition 3.1.2. □

5.4 Dedekind-η Products

In this section we review a construction that produces scalar-valued modular forms over $\widetilde{\Gamma_0(N)}$. The Dedekind-η function is given by

$$\eta(\tau) = \mathbf{q}^{1/24}\prod_{k=1}^{\infty}(1-\mathbf{q}^k) \qquad (5.42)$$

and is a weight $\frac{1}{2}$ modular form on $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$. It satisfies

$$\eta(\tau+1) = \mathbf{e}(1/12)\eta(\tau), \tag{5.43}$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau). \tag{5.44}$$

Let $\eta_m(\tau) = \eta(m\tau)$.

Theorem 5.4.1 (Theorem 6.2 of [3]). *Given the following*

1) a lattice L with the level of Λ_L equal to N,

2) r_δ for $\delta \mid N$ such that $|\Lambda_L|/\prod_{\delta|N}\delta^{r_\delta}$ is a rational square,

3) $(1/24)\sum_{\delta|N} r_\delta\delta \in \mathbb{Z}$, and

4) $(N/24)\sum_{\delta|N} r_\delta/\delta \in \mathbb{Z}$,

then $\prod\limits_{\delta|N}\eta_\delta^{r_\delta}$ is a modular form for $\widetilde{\Gamma_0(N)}$ of weight $k = \sum_\delta r_\delta/2$ and of character $\chi_{|\Lambda_L|}$ if $4 \nmid N$ and $\chi_\theta^{2k+\left(\frac{-1}{|\Lambda_L|}\right)-1}\chi_{2^{2k}|\Lambda_L|}$ if $4 \mid N$.

Chapter 6

Calculating the $\kappa_\eta(m)$

The application of Theorem 4.5.1 requires summing over $\lambda \in L/(L_+ + L_-)$ and also computing $\kappa^-_{\eta_- + \lambda_-}(m)$. This chapter will cover the general techniques to complete these two tasks.

6.1 The Structure of $L/(L_+ + L_-)$

Let $\mathcal{O}$, m, and q be as in Proposition 2.2.1. Then a straightforward calculation yields

$$L = \mathcal{O} \cap V = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3 \tag{6.1}$$

where

$$\ell_1 = \alpha, \qquad \ell_2 = \tfrac{m\alpha + \alpha\beta}{q}, \qquad \ell_3 = \tfrac{\beta + \alpha\beta}{2}. \tag{6.2}$$

Fix a squarefree $d \in \mathbb{Z}$ and an element $z = z_1\ell_1 + z_2\ell_2 + z_3\ell_3$ with

$$\begin{aligned} d &= Q(z) && (6.3)\\ &= -qz_1^2 - 2mz_1z_2 - \left(\frac{m^2 - D}{q}\right) z_2^2 + Dz_2z_3 + \left(\frac{q-1}{4}\right) Dz_3^2. && (6.4) \end{aligned}$$

Recall that $D \equiv m^2 \mod q$, $2D \mid m$, and $q \equiv 5 \mod 8$, hence each summand is integral.

To ease notation in the following, define the following quantities.

$$\alpha_1 = D(z_2 + (q-1)z_3/2), \tag{6.5}$$

$$\alpha_2 = 2(qz_1 + mz_2), \tag{6.6}$$

$$\alpha_3 = -2(D - m^2)z_2/q + 2mz_1 - Dz_3, \tag{6.7}$$

$$\gamma_{123} = \gcd(\alpha_1, \alpha_2, \alpha_3). \tag{6.8}$$

Lemma 6.1.1. *If* $2 \mid dD$*, then* $\gamma_{123} = \gcd(D, 2d)$.

Proof. Set

$$\begin{aligned} m &= 2Dk_1, \\ D - 4D^2k_1^2 &= Dqk_2. \end{aligned} \tag{6.9}$$

Consider the integral matrix

$$W = \begin{pmatrix} q & -Dk_1(q-1) & q(q-1)/2 \\ -4k_1 & 4Dk_1^2 + k_2 & -2k_1(q-1) \\ -2 & 2Dk_1 & -q \end{pmatrix}, \tag{6.10}$$

which has determinant $\det(W) = -4Dk_1^2 - qk_2 = -1$. Then

$$\begin{aligned} \gamma_{123} &= \gcd(\alpha_1, \alpha_2, \alpha_3) \\ &= \gcd\Big(q\alpha_1 - Dk_1(q-1)\alpha_2 + q\frac{(q-1)}{2}\alpha_3, \\ &\qquad -4k_1\alpha_1 + (4Dk_1^2 + k_2)\alpha_2 - 2k_1(q-1)\alpha_3, -2\alpha_1 + 2Dk_1\alpha_2 - q\alpha_3\Big) \\ &= \gcd(Dz_2, 2z_1, Dz_3). \end{aligned}$$

Now consider two cases, keeping in mind that both d and D are squarefree and hence the z_i are relatively prime.

Case 1: $2 \mid D$. Since $2 \mid D$, $2 \mid \gamma_{123}$ and $2 \mid \gcd(D, 2d)$. So consider only odd primes p. If $p \mid \gamma_{123}$ then $p \mid z_1$ and $p \mid D$. By (6.9), the expression in (6.4) becomes

$$d = -qz_1^2 - 4Dk_1z_1z_2 + Dk_2z_2^2 + Dz_2z_3 + \left(\frac{q-1}{4}\right) Dz_3^2. \tag{6.11}$$

Thus $p \mid d$ and $\gamma_{123} \mid \gcd(D, 2d)$. Now suppose $p \mid \gcd(D, 2d)$, then again by (6.11) $p \mid 2qz_1^2$. Since p is odd and $q \nmid D$, $p \mid z_1$. Thus $\gcd(D, 2d) \mid \gamma_{123}$.

Case 2: $2 \mid d$**,** $2 \nmid D$. Since $2 \nmid D$, if $2 \mid \gamma_{123}$, then $2 \mid z_2$ and $2 \mid z_3$. But then by (6.11), $2 \mid z_1$ which is a contradiction. So $2 \nmid \gamma_{123}$. Then the same argument for odd primes p as in Case 1 gives $\gamma_{123} = \gcd(D, d) = \gcd(D, 2d)$. □

Note. *If* $2 \nmid dD$*, then* $\gcd(D, 2d) \mid \gamma_{123} \mid 2\gcd(D, 2d)$ *and equality does depend on the choice of the* z_i*.*

Let $U = z^{\perp} \subset V$ be the negative plane that is perpendicular to z. A straightforward computation yields an orthogonal basis for U given by

$$\begin{aligned} u_1 &= (2q\alpha_1)\ell_2 + (2q\alpha_3)\ell_3, \\ u_2 &= \left(\frac{4\alpha_1^2(m^2 - D) - 4Dq\alpha_1\alpha_3 - D(q-1)q\alpha_3^2}{2D}\right)\ell_1 \\ &\quad - \left(\frac{q(2D\alpha_1\alpha_2 - 4m\alpha_1^2 + D(q-1)\alpha_2\alpha_3)}{2D}\right)\ell_2 \\ &\quad + \left(\frac{2\alpha_1\alpha_2(D - m^2) + q\alpha_3(D\alpha_2 - 2m\alpha_1)}{D}\right)\ell_3. \end{aligned}$$

Another direct calculation shows

$$Q(xu_1 + yu_2) = \mathcal{C}(x^2 + dy^2), \tag{6.12}$$

where the constant is given by

$$\mathcal{C} = q(4\alpha_1^2(Dm^2) + 4Dq\alpha_1\alpha_3 + Dq(q-1)\alpha_3^2).$$

Thus U can be viewed as $Q(\sqrt{-d})$ with a quadratic form given by a constant times the field norm.

Recall that L has the following subspaces,

$$L_- = L \cap U, \quad (6.13)$$

$$L_+ = L \cap \mathbb{Q}z. \quad (6.14)$$

Theorem 6.1.2. *$L/(L_- + L_+)$ is a finite group of order $\frac{2d}{\gamma_{123}}$.*

Proof. The proof relies mostly on the following lemma. Define

$$\begin{aligned} \gamma_{12} &= \tfrac{\gcd(\alpha_1,\alpha_2)}{\gamma_{123}}, \quad \alpha_1' = \tfrac{\alpha_1}{\gamma_{12}\gamma_{13}\gamma_{123}}, \\ \gamma_{13} &= \tfrac{\gcd(\alpha_1,\alpha_3)}{\gamma_{123}}, \quad \alpha_2' = \tfrac{\alpha_2}{\gamma_{12}\gamma_{23}\gamma_{123}}, \\ \gamma_{23} &= \tfrac{\gcd(\alpha_2,\alpha_3)}{\gamma_{123}}, \quad \alpha_3' = \tfrac{\alpha_3}{\gamma_{13}\gamma_{23}\gamma_{123}}. \end{aligned} \quad (6.15)$$

Lemma 6.1.3. *A basis for L_- is given as*

$$\begin{aligned} \ell_1^- &= \alpha_1'\gamma_{12}\ell_2 + \alpha_3'\gamma_{23}\ell_3, \\ \ell_2^- &= \gamma_{13}\ell_1 + s_1\alpha_2'\gamma_{12}\ell_2 - \alpha_2'\gamma_{23}r_1\ell_3, \end{aligned} \quad (6.16)$$

where $r_1\alpha_1' + s_1\alpha_3' = 1$.

Proof. Suppose that $ru_1 + su_2 \in L$ for $r, s \in \mathbb{Q}$. Looking at the ℓ_1-component of $ru_1 + su_2$ gives

$$s = \frac{-Ds'}{q(4\alpha_1^2(D - m^2) + 4Dq\alpha_1\alpha_3 + Dq(q-1)\alpha_3^2)} \quad (6.17)$$

for some $s' \in \mathbb{Z}$. Substituting this back into $ru_1 + su_2$ and considering the ℓ_2-component gives r equal to

$$\frac{4\alpha_1(r'\alpha_1 - \alpha_2 s')(D - m^2) + q\alpha_3(Dr'(4\alpha_1 - \alpha_3) + 2s'(\alpha_2 D - 2\alpha_1 m)) + r'\alpha_3^2 Dq^2}{4\alpha_3 q^2(4\alpha_1^2(D - m^2) + 4\alpha_1\alpha_3 Dq + \alpha_3^2 D(q-1)q}$$

for some $r' \in \mathbb{Z}$. Finally, considering the ℓ_3-component gives

$$r'\alpha_1 \quad \equiv \quad s'\alpha_2 \quad \mod \alpha_3. \tag{6.18}$$

Thus

$$r'\frac{\alpha_1}{\gamma_{123}} \quad \equiv \quad s'\frac{\alpha_2}{\gamma_{123}} \quad \mod \frac{\alpha_3}{\gamma_{123}}. \tag{6.19}$$

Set $r' = \gamma_{23}r''$ and $s' = \gamma_{13}s''$ for some $r'', s'' \in \mathbb{Z}$. This gives

$$r''\alpha_1' \equiv s''\alpha_2' \quad \mod \alpha_3'. \tag{6.20}$$

Therefore, choose $r_1, s_1 \in \mathbb{Z}$ such that $r_1\alpha_1' + s_1\alpha_3' = \gcd(\alpha_1', \alpha_3') = 1$. Then

$$r'' = s''r_1\alpha_2' + \alpha_3't'' \tag{6.21}$$

for some $t'' \in \mathbb{Z}$. Plugging all of these substitutions in yields the basis given by (6.16). □

Returning to the proof of the theorem, consider the relation matrix for $L/(L_- + L_+)$,

$$M = \begin{pmatrix} z_1 & z_2 & z_3 \\ \gamma_{13} & s_1\alpha_2'\gamma_{12} & -\alpha_2'\gamma_{23}r_1 \\ 0 & \alpha_1'\gamma_{12} & \alpha_3'\gamma_{23} \end{pmatrix}. \tag{6.22}$$

This matrix has $\det(M) = \frac{2d}{\gamma_{123}}$, thus $L/(L_- + L_+)$ has order $\frac{2d}{\gamma_{123}}$. □

Corollary 6.1.4. *If $2 \mid D$ or $2 \nmid d$, then $L/(L_- + L_+)$ is cyclic.*

Proof. The relation matrix can be put into Smith normal form by elementary row and column operations,

$$E_{\text{row}}ME_{\text{column}} = \text{diag}(\sigma_1, \sigma_2, \sigma_3), \tag{6.23}$$

where $\sigma_1 \mid \sigma_2 \mid \sigma_3$ and $\sigma_1\sigma_2\sigma_3 = \frac{2d}{\gamma_{123}}$. Under the given conditions, $\frac{2d}{\gamma_{123}}$ is squarefree, thus $\sigma_1 = \sigma_2 = 1$ and $L/(L_- + L_+)$ is cyclic. □

Note. *When* $\frac{2d}{\gamma_{123}}$ *is not squarefree,*

$$L/(L_- + L_+) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/\left(\frac{d}{\gamma_{123}}\right)\mathbb{Z}$$

and this may or may not be cyclic.

6.2 Eisenstein Series and Whittaker Polynomials

This section summarizes the notation and results from [10] and [13]. Recall that for $\mu \in L_-^\vee/L_-$ and $\psi_\mu = char(\mu + L_-)$,

$$E(\tau, s; \psi_\mu, +1) = \sum_{m\in\mathbb{Q}} A_\mu(s, m, v)\mathbf{q}^m = \sum_{m\in\mathbb{Q}} E_m(\tau, s, \mu) \tag{6.24}$$

where $\tau = u + iv$. The Fourier coefficients have Laurant expansions

$$A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2) \tag{6.25}$$

at $s = 0$. Thus

$$b_\mu(m, v) = \frac{\partial}{\partial s}\left\{A_\mu(s, m, v)\right\}_{s=0} \tag{6.26}$$

$$= \mathbf{q}^{-m}\frac{\partial}{\partial s}\left\{E_m(\tau, s, \mu)\right\}_{s=0}. \tag{6.27}$$

Let Δ denote the discriminant of $\boldsymbol{k} \simeq U$ and $h(\boldsymbol{k})$ its ideal class number. Following [10], there is a normalization which satisfies

$$h(\boldsymbol{k})\frac{\partial}{\partial s}\left\{E_m(\tau, s, \mu)\right\}_{s=0} = \frac{\partial}{\partial s}\left\{E_m^*(\tau, s, \mu)\right\}_{s=0} \tag{6.28}$$

and a factorization of $E^*_m(\tau,s,\mu)$ into Whittaker polynomials,

$$E^*_m(\tau,s,\mu) = v^{-\frac{1}{2}}|\Delta|^{\frac{s+1}{2}} W^*_{m,\infty}(\tau,s,\mu)\prod_p W^*_{m,p}(s,\mu). \tag{6.29}$$

Proposition 6.2.1 ([10]). *The following values are obtained at $s=0$.*

1) $E^*_m(\tau,0,\mu)=0$.

2) $W^*_{m,\infty}(\tau,0,\mu) = -\gamma_\infty 2v^{\frac{1}{2}}\mathbf{q}^m$,

where γ_∞ is a local factor that will not affect later calculations since $\prod_{p\leq\infty}\gamma_p = 1$.

Corollary 6.2.2. *There exists a finite prime p' such that $W^*_{m,p'}(0,\mu)=0$ and hence*

$$b_\mu(m,v) = \frac{-2\sqrt{|\Delta|}\gamma_\infty}{h(\boldsymbol{k})}\frac{\partial}{\partial s}\left\{W^*_{m,p'}(s,\mu)\right\}_{s=0}\prod_{p\neq p'} W^*_{m,p}(0,\mu). \tag{6.30}$$

Note that in this case $b_\mu(m,v)$ does not depend on v and thus (6.30) is equal to the limit in (4.27).

Now [13] provides the method to compute $W^*_{m,p}(s,\mu)$. Set

$$W_{m,p}(s,\mu) = \frac{W^*_{m,p}(s,\mu)}{L_p(s+1,\chi_\Delta)}.$$

Let S be the matrix for the quadratic form on the lattice L. As usual, address the cases of $p=2$ and p odd separately.

In the case of p odd, S is $\mathbb{Z}_p$-equivalent to a diagonal matrix, $\mathrm{diag}(\epsilon_1 p^{l_1}, \epsilon_2 p^{l_2})$, with $\epsilon_i \in \mathbb{Z}_p^\times$, $l_i\in\mathbb{Z}$. Then for $\mu=(\mu_1,\mu_2)\in L_p^\vee$, $\alpha\in\mathbb{Z}_p^\times$, $a\in\mathbb{Z}$ and $0<k\in\mathbb{Z}$,

define the following quantities.

$$
\begin{aligned}
H_\mu(k) &= \{i \mid \mu_i \in \mathbb{Z}_p\}, && (6.31)\\
K_\mu &= \begin{cases} \infty & \text{if } \mu \in L \\ \min_{i \notin H_\mu}(l_i + \mathrm{ord}_p \mu_i) & \text{if } \mu \notin L \end{cases}, && (6.32)\\
L_\mu(k) &= \{i \in H_\mu \mid l_i - k < 0 \text{ and is odd }\}, && (6.33)\\
l_\mu(k) &= \#L(k), && (6.34)\\
d_\mu(k) &= k + \frac{1}{2}\sum_{\substack{l_i<k \\ i\in H_\mu}} (l_i - k), && (6.35)\\
v_\mu(k) &= \left(\frac{-1}{p}\right)^{[\frac{l(k)}{2}]} \prod_{i \in L(k)} \left(\frac{\epsilon_i}{p}\right), && (6.36)\\
f(\alpha p^a) &= \begin{cases} \frac{-1}{p} & \text{if } l(a+1) \text{ is even} \\ \left(\frac{\alpha}{p}\right)\frac{1}{\sqrt{p}} & \text{if } l(a+1) \text{ is odd} \end{cases}, && (6.37)\\
t_\mu(m) &= m - \sum_{i \notin H_\mu} \epsilon_i p^{l_i} \mu_i^2, && (6.38)
\end{aligned}
$$

where $[\frac{l(k)}{2}]$ is the integer part of $\frac{l(k)}{2}$.

Theorem 6.2.3 ([13]). *$W_{m,p}(s,\mu) = 0$ unless $m \in Q(\mu) + \mathbb{Z}_p$. In such a case, let $X = p^{-s}$ and $a = ord_p(t_\mu(m))$. Then*

$$\frac{W_{m,p}(s,\mu)}{\gamma_p|\det 2S|_p^{\frac{1}{2}}} = 1+(1-\frac{1}{p})\sum_{\substack{0<k\leq a \\ l_\mu(k)\ even}} v_\mu(k)p^{d_\mu(k)}X^k + v_\mu(a+1)f(t_\mu(m))p^{d_\mu(a+1)}X^{a+1}$$

when $0 \leq a < K_\mu$, and

$$\frac{W_{m,p}(s,\mu)}{\gamma_p|\det 2S|_p^{\frac{1}{2}}} = 1+(1-\frac{1}{p})\sum_{\substack{0<k\leq K_\mu \\ l_\mu(k)\ even}} v_\mu(k)p^{d_\mu(k)}X^k \qquad (6.39)$$

when $a \geq K_\mu$.

Corollary 6.2.4. *If* $K_\mu = 0$, *then* $W^*_{m,p}(s,\mu) = Char(Q(\mu)+\mathbb{Z}_p)(m)$.

Corollary 6.2.5. *Suppose* $U \simeq \boldsymbol{k}$ *with discriminant* Δ. *If* L *is unimodular (self-dual) and* $m \in Q(\mu)+\mathbb{Z}_p$, *then*

$$W^*_{m,p}(s,\mu) = \sum_{r=0}^{ord_p(m)} \left(\frac{\Delta}{p}\right)^r X^r. \tag{6.40}$$

Thus

$$W^*_{m,p}(0,\mu) = \rho_p(m) = \sum_{r=0}^{ord_p(m)} \left(\frac{\Delta}{p}\right)^r. \tag{6.41}$$

If $\rho_p(m) = 0$, *then*

$$W^{*,'}_{m,p}(0,\mu) = \frac{1}{2}\log(p)(ord_p(m)+1)\rho_p(m/p). \tag{6.42}$$

Note that $\rho_p(m) = \rho_p(p^{\mathrm{ord}_p(m)})$, and $\rho_p(1) = 1$. Hence $W^*_{m,p}(0,\mu) \neq 1$ for only a finite number of primes and (6.29) is a finite product.

For the $p = 2$ case, let ψ be the canonical additive character on $\mathbb{Q}_p/\mathbb{Z}_p$. Also recall the character $\left(\frac{2}{x}\right)$ defined by (5.12). Now the matrix S is $\mathbb{Z}_2$-equivalent to only one of the following,

$$\begin{pmatrix} \epsilon_1 2^{l_1} & \\ & \epsilon_2 2^{l_2} \end{pmatrix}, \tag{6.43}$$

$$\epsilon' 2^{l'} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \tag{6.44}$$

$$\epsilon'' 2^{l''} \begin{pmatrix} & \frac{1}{2} \\ \frac{1}{2} & \end{pmatrix}, \tag{6.45}$$

for ϵ_i, ϵ', $\epsilon'' \in \mathbb{Z}_2^\times$ and l_i, l', $l'' \in \mathbb{Z}$.

We will now specialize a theorem from [13] in these three cases. First consider $S \sim (6.43)$. Then for $\mu = (\mu_1, \mu_2) \in L_2^\vee$ and $0 < k \in \mathbb{Z}$ define the following quantities.

$$
\begin{aligned}
H_\mu &= \{i \mid \mu_i \in \mathbb{Z}_2\}, && (6.46)\\
L_\mu(k) &= \{i \in H_\mu \mid l_i - k < 0 \text{ is odd }\}, && (6.47)\\
l_\mu(k) &= \# L_\mu(k), && (6.48)\\
d_\mu(k) &= k + \frac{1}{2}\sum_{i \in H_\mu} \min(l_i - k + 1, 0), && (6.49)\\
\epsilon_\mu(k) &= \prod_{i \in L_\mu(k-1)} \epsilon_i, && (6.50)\\
\delta_\mu(k) &= \begin{cases} 0 & \text{if } l_i = k-1 \text{ for some } i \in H_\mu \\ 1 & \text{otherwise.} \end{cases}, && (6.51)\\
Q'(\mu) &= \sum_{i \notin H_\mu} \epsilon_i 2^{l_i} \mu_i^2, && (6.52)\\
\nu = \nu_\mu(m,k) &= (m - Q'(\mu))2^{3-k} - \sum_{i \in H_\mu,\, l_i < k-1} \epsilon_i. && (6.53)
\end{aligned}
$$

Let K_μ be ∞ if $\mu \in L_2$. For $\mu \in L_2^\vee - L_2$, let K_μ be the minimum of

$$\{l_i + \mathrm{ord}_2\mu_i + 1 \mid \mathrm{ord}_2\mu_i < -1\}, \qquad \{l_i + 1 \mid \mathrm{ord}_2\mu_i = -1\}. \tag{6.54}$$

Theorem 6.2.6 ([13]). *For $S \sim$ (6.43) and $\mu \in L_2^\vee$,*

$$
\begin{aligned}
\frac{2^{\left(\frac{l_1+l_2+2}{2}\right)} W_{m,2}(s,\mu)}{\gamma_2} &= Char(Q(\mu)+\mathbb{Z}_2)(m) \\
&+ \sum_{\substack{0<k\le K_\mu \\ l_\mu(k)\ odd}} \delta_\mu(k) 2^{d_\mu(k)-3/2} \left(\frac{2}{\epsilon_\mu(k)\nu}\right) X^k \\
&+ \sum_{\substack{0<k\le K_\mu \\ l_\mu(k)\ even}} \delta_\mu(k) 2^{d_\mu(k)-1} \left(\frac{2}{\epsilon_\mu(k)}\right) \psi\left(\frac{\nu}{8}\right) Char(4\mathbb{Z}_2)(\nu) X^k.
\end{aligned}
$$

Note that even when $K_\mu = \infty$, both sums are finite since there always exists $K^{\max}$ such that for all $k > K^{\max}$, $\mathrm{Char}(4\mathbb{Z}_2)(\nu(k)) = 0$ and $\left(\frac{2}{\epsilon_\mu(k)\nu(k)}\right) = 0$.

Next consider $S \sim$ (6.44). (Re)define for $\mu = (\mu_1, \mu_2) \in L_2^\vee$ and $0 < k \in \mathbb{Z}$ the following quantities.

$$K_\mu = \begin{cases} \infty & \text{if } \mu \in L_2 \\ l' + \min(\mathrm{ord}_2(\mu_1), \mathrm{ord}_2(\mu_2)) & \text{if } \mu \notin L_2 \end{cases}, \tag{6.55}$$

$$d_\mu(k) = \begin{cases} \min(l', k) & \text{if } \mu \in L_2 \\ k & \text{if } \mu \notin L_2 \end{cases}, \tag{6.56}$$

$$p_\mu(k) = \begin{cases} (-1)^{\min(l'-k,0)} & \text{if } \mu \in L_2 \\ 1 & \text{if } \mu \notin L_2 \end{cases}, \tag{6.57}$$

$$Q'(\mu) = \begin{cases} 0 & \text{if } \mu \in L_2 \\ Q(\mu) & \text{if } \mu \notin L_2 \end{cases}, \tag{6.58}$$

$$\nu(k) = \nu_\mu(m, k) = (m - Q'(\mu))2^{3-k}. \tag{6.59}$$

Theorem 6.2.7 ([13]). *For $S \sim$ (6.44) and $\mu \in L_2^\vee$,*

$$\frac{2^{l'} W_{m,2}(s,\mu)}{\gamma_2} = Char(Q(\mu) + \mathbb{Z}_2)(m) + \sum_{0<k\le K_\mu} p_\mu(k) 2^{d_\mu(k)-1} \psi\left(\frac{\nu(k)}{8}\right) Char(4\mathbb{Z}_2)(\nu(k)) X^k.$$

Lastly, for $S \sim$ (6.45). Redefine for $\mu = (\mu_1, \mu_2) \in L_2^\vee$ and $0 < k \in \mathbb{Z}$ the following quantities.

$$K_\mu = \begin{cases} \infty & \text{if } \mu \in L_2 \\ l'' + \min(\mathrm{ord}_2(\mu_1), \mathrm{ord}_2(\mu_2)) & \text{if } \mu \notin L_2 \end{cases}, \tag{6.60}$$

$$d_\mu(k) = \begin{cases} \min(l'', k) & \text{if } \mu \in L_2 \\ k & \text{if } \mu \notin L_2 \end{cases}, \tag{6.61}$$

$$Q'(\mu) = \begin{cases} 0 & \text{if } \mu \in L_2 \\ Q(\mu) & \text{if } \mu \notin L_2 \end{cases}, \tag{6.62}$$

$$\nu(k) = \nu_\mu(m, k) = (m - Q'(\mu))2^{3-k}. \tag{6.63}$$

Theorem 6.2.8 ([13]). *For $S \sim$ (6.45) and $\mu \in L_2^\vee$,*

$$\frac{2^{l''} W_{m,2}(s,\mu)}{\gamma_2} = Char(Q(\mu) + \mathbb{Z}_2)(m) + \sum_{0<k\leq K_\mu} 2^{d_\mu(k)-1}\psi\left(\frac{\nu(k)}{8}\right) Char(4\mathbb{Z}_2)(\nu(k))X^k.$$

Through (6.30), Theorems 6.2.6-6.2.8 can now be used to explicitly calculate $\kappa_u^-(m)$.

Chapter 7

Examples

In [5], Elkies demonstrates a technique for computing the coordinates of some rational CM points on the Shimura curves associated to quaternion algebras with small discriminants. We now review the set-up from [5] in the cases of $D = 6$ and $D = 10$ using the notations and conventions introduced in the previous chapters.

It should be noted that in [5] the cases of $D = 14$ and $D = 15$ are also considered. For $D = 14$, the exponents on only the odd primes in the factorizations of rational CM points could be verified with our method. This is likely the result of a small systematic error in the explicit local calculations at the prime 2, but the general theory should still apply. In the case of $D = 15$, the correct expression in terms of rational quadratic divisors for the divisor of the coordinate map has not yet been computed.

7.1 $D = 6$

First consider the quaternion algebra ramified at the primes 2 and 3. In [5] it is presented as $\mathbb{Q} + \mathbb{Q}b + \mathbb{Q}c + \mathbb{Q}bc$ with

$$b^2 - 2 = c^2 + 3 = bc + cb = 0. \tag{7.1}$$

By Theorem 2.1.1, there is a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}+\mathbb{Q}b+\mathbb{Q}c+\mathbb{Q}bc \xrightarrow{\sim} B = \left(\frac{5,6}{\mathbb{Q}}\right)$. In this case it is given by

$$b \mapsto \frac{4}{5}\alpha - \frac{1}{5}\alpha\beta, \tag{7.2}$$

$$c \mapsto -\frac{3}{5}\alpha + \frac{2}{5}\alpha\beta, \tag{7.3}$$

$$bc \mapsto \beta. \tag{7.4}$$

The algebra B has a maximal order given by

$$\mathcal{O} = \mathbb{Z} + \left(\frac{4\alpha - \alpha\beta}{5}\right)\mathbb{Z} + \left(\frac{5 - 3\alpha + 2\alpha\beta}{10}\right)\mathbb{Z} + \left(\frac{4\alpha + 5\beta - \alpha\beta}{10}\right)\mathbb{Z}. \tag{7.5}$$

Further, the image of Γ^* in $\mathrm{PGL}_2(\mathbb{R})$ is generated by three elements,

$$s_2 = -\frac{6}{5}\alpha + \beta + \frac{4}{5}\alpha\beta, \tag{7.6}$$

$$s_4 = 1 - \frac{1}{5}\alpha + \frac{1}{2}\beta + \frac{3}{10}\alpha\beta, \tag{7.7}$$

$$s_6 = \frac{3}{2} - \frac{3}{10}\alpha + \frac{1}{5}\alpha\beta, \tag{7.8}$$

which satisfy the group presentation

$$< s_2, s_4, s_6 \mid s_2^2 = s_4^4 = s_6^s = s_2 s_4 s_6 = 1 > \tag{7.9}$$

(See Section 3.1 of [5]). Since the image of Γ^* has such a presentation, it is called a triangle group, and there exists a parametrization $t_6 : \mathcal{X}_6^* \xrightarrow{\sim} \mathbb{P}^1$ over $\mathbb{Q}$. Such a map giving the isomorphism is only well-defined up to a PGL_2 action on $\mathbb{P}^1$. However, the map is uniquely determined once the value at three points of $\mathcal{X}_6^*$ are choosen. Since there are three distinguished elements of Γ^*, namely s_2, s_4, s_6, it is only natural to fix the value of the isomorphism at their three fixed points, P_2, P_4, P_6. Thus, define the map $t_6 : \mathcal{X}_6^* \xrightarrow{\sim} \mathbb{P}^1$ such that it takes on the values 0, 1, ∞ at the points P_4, P_2, P_6, respectively. (Warning: In [5], the author chooses t_6 to have the values 0, 1, ∞ at the points P_2, P_4, P_6.) This defining criteria can be expressed as

$$\begin{aligned} \operatorname{div}(t_6) &= P_4 - P_6, \\ t_6(P_2) &= 1. \end{aligned} \tag{7.10}$$

Let s_i^0 denote the trace-0 part of s_i. Since the action of $B^\times$ factors through $\mathrm{PGL}_2(\mathbb{R})$, the fixed point of s_i is the fixed point of all of $\boldsymbol{k}_i^\times \subset B^\times$ where $\boldsymbol{k}_i = \mathbb{Q}(s_i) = \mathbb{Q}(s_i^0)$. For the s_i as above,

$$\boldsymbol{k}_2 \simeq \mathbb{Q}(\sqrt{-6}), \qquad \boldsymbol{k}_4 \simeq \mathbb{Q}(\sqrt{-1}), \qquad \boldsymbol{k}_6 \simeq \mathbb{Q}(\sqrt{-3}). \tag{7.11}$$

Then Proposition 4.2.1 and Definition 4.2.1 suggest expressing $\operatorname{div}(t_6)$ as a linear combination of $Z(1, \eta_1; \Gamma^*)$ and $Z(3, \eta_3; \Gamma^*)$ for some $\eta_1, \eta_3 \in L^\vee/L$.

Lemma 7.1.1. *The stabilizers of the s_i^0 have the following sizes.*

1) $|Stab_{\Gamma^*}(s_4^0)| = 4$.

2) $|Stab_{\Gamma^*}(s_6^0)| = 6$.

Proof. To avoid redundancy, we will only calculate $|\mathrm{Stab}_{\Gamma^*}(s_6^0)|$ explicitly. The order of $\mathrm{Stab}_{\Gamma^*}(s_4^0)$ is calculated in the same fashion. A routine calculation shows that $\mathrm{Stab}_{B^\times}(x) = (\mathbb{Q} + \mathbb{Q}x) \cap B^\times$. However, by Section 3.5 of [12], $r + ts_6^0 \in \Gamma^*$ for $r, t \in \mathbb{Q}$ if and only if $\mathrm{n}(r + ts_6^0) = r^2 + 4t^2 \in \mathbb{Z}_p^\times$ for $p \nmid D$. Since the Γ^* action factors through $\mathrm{PGL}_2(\mathbb{R})$, assume that $r, t \in \mathbb{Z}$ and are relatively prime. Suppose

$$r^2 + 3t^2 = 2^j u_2 \tag{7.12}$$

for some odd $u_2 \in \mathbb{Z}$. Then either $r = 0$, $t = 0$, or r and t are odd. If $r = 0$ or $t = 0$, then $j = 0$ or 2. Suppose $r = 2r' + 1$ and $t = 2t' + 1$ for some $r', t' \in \mathbb{Z}$. Then $4(r^2 + r) + 1 + 12(t^2 + t) + 3 = 2^j u_2$. Thus $j = 2$. Now suppose

$$r^2 + 3t^2 = 3^k u_3 \tag{7.13}$$

for some $u_3 \in \mathbb{Z}$ relatively prime to 3. Either $k = 0$ or $3 \mid m$. Suppose $3 \mid r$. Then $3(r/3)^2 + t^2 = 3^{k-1}u_3$. Since r and t were relatively prime, $k = 1$. Thus there are only four possible values that $r^2 + 3t^2$ can take on for $r + ts_6^0 \in \Gamma^*$: 1, 3, 4, and 12. Inspection reveals that $(r, t) \in \{(1, 0), (0, 1), (1, \pm 1), (3, \pm 1)\}$. Hence $|\mathrm{Stab}_{\Gamma^*}(s_6^0)| = 6$. □

Lemma 7.1.2. *The following equalities hold.*

1) $Z(1, 0; \Gamma^*) = \frac{1}{4} P_4$.

2) $Z(3, 0; \Gamma^*) = \frac{1}{6} P_6$.

Proof. These identities follow directly from Lemma 7.1.1 and that

$$|\Gamma^* \backslash L(1)| = |\Gamma^* \backslash L(3)| = 1 \tag{7.14}$$

by Corollary 3.2.10. □

Proposition 7.1.3.

$$div(t_6) \quad = \quad 4Z(1, 0; \Gamma^*) - 6Z(3, 0; \Gamma^*). \tag{7.15}$$

Hence, to use Theorem 4.3.1, the input vector-valued form must have, for $m > 0$,

$$c_0(-m) = \begin{cases} 2 & m = 1 \\ -3 & m = 3 \\ 0 & \text{otherwise} \end{cases} . \tag{7.16}$$

7.1.1 The Input Form

Recall that $\Lambda_L = L^\vee/L$. By Corollary 3.1.1, $|\Lambda_L| = 72$ and $N = 12$. To vectorize properly, we need a form of weight $\frac{1}{2}$ and character $\chi_\theta\chi_{144}$.

Proposition 7.1.4. *Let $A_1, A_2, A_3, A_4, A_5 \in \mathbb{Z}$, and set*

$$\begin{aligned} r_1 &= A_5, && (7.17)\\ r_2 &= 16 - 12A_1 + 36A_2 - 9A_3 - 14A_4 - 6A_5, && (7.18)\\ r_3 &= -30 + 24A_1 - 48A_2 + 16A_3 + 24A_4 + 5A_5, && (7.19)\\ r_4 &= -17 + 12A_1 - 36A_2 + 9A_3 + 16A_4 + 5A_5, && (7.20)\\ r_6 &= 43 - 36A_1 + 60A_2 - 21A_3 - 34A_4 - 6A_5, && (7.21)\\ r_{12} &= -11 + 12A_1 - 12A_2 + 5A_3 + 8A_4 + A_5. && (7.22) \end{aligned}$$

Then

$$\prod_{\delta|12} \eta_\delta^{r_\delta} \tag{7.23}$$

is a modular form for $\widetilde{\Gamma_0(12)}$ of weight $\frac{1}{2}$ and of character $\chi_\theta\chi_{144}$.

Proof. One can check that the following hold.

$$\begin{aligned} 72/\prod_{\delta|12} \delta^{r_\delta} &= (2^{A_3}3^{A_4})^2, && (7.24)\\ (1/24)\sum_{\delta|12} r_\delta \delta &= A_1, && (7.25)\\ (1/2)\sum_{\delta|12} r_\delta/\delta &= A_2, && (7.26)\\ \sum_{\delta} r_\delta/2 &= \frac{1}{2}. && (7.27) \end{aligned}$$

Hence, by the Theorem 5.4.1, (7.23) is a modular form for $\widetilde{\Gamma_0(12)}$ of weight $\frac{1}{2}$ and of character $\chi_\theta \chi_{144}$. □

Now examine the structure of such a form at the various cusps of $\widetilde{\Gamma_0(12)}$. The orders of the zeroes of η_δ at the cusps for $\delta \mid N$ are given in Table 7.1. Using this information, Table 7.2 gives the orders of the zeroes for a form defined by (7.17-7.23), where a negative value represents a pole.

To construct a form defined by (7.17-7.23) such that it has neither a pole nor a zero at $i\infty$ and no pole at any other cusp, one simply solves the following system of inequalities over $\mathbb{Z}$.

$$0 \leq A_2/12, \tag{7.28}$$

$$0 \leq (15 - 12A_1 + 28A_2 - 8A_3 - 12A_4 - 4A_5)/12, \tag{7.29}$$

$$0 \leq (-5 + 4A_1 - 9A_2 + 3A_3 + 4A_4 + A_5)/4, \tag{7.30}$$

$$0 \leq (-4 + 3A_1 - 8A_2 + 2A_3 + 4A_4 + A_5)/3, \tag{7.31}$$

$$0 \leq (25 - 20A_1 + 36A_2 - 12A_3 - 20A_4 - 4A_5)/4, \tag{7.32}$$

$$0 = A_1, \tag{7.33}$$

Doing so yields a unique solution

$$(A_1, A_2, A_3, A_4, A_5) = (0, 0, 1, 1, -2) \tag{7.34}$$

which produces

$$\psi_0 = \frac{\eta_2^5}{\eta_1^2 \eta_4^2} = \theta(\tau) = \sum_{n \in \mathbb{Z}} \mathbf{q}^{n^2}. \tag{7.35}$$

Table 7.1: Order of the zero of η_δ at the cusps of $\widetilde{\Gamma_0(12)}$

	Cusp					
	$1=0$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}=i\infty$
η	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
η_2	$\frac{1}{48}$	$\frac{1}{12}$	$\frac{1}{48}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
η_3	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{8}$	$\frac{1}{72}$	$\frac{1}{8}$	$\frac{1}{8}$
η_4	$\frac{1}{96}$	$\frac{1}{24}$	$\frac{1}{96}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{6}$
η_6	$\frac{1}{144}$	$\frac{1}{36}$	$\frac{1}{16}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{1}{4}$
η_{12}	$\frac{1}{288}$	$\frac{1}{72}$	$\frac{1}{32}$	$\frac{1}{18}$	$\frac{1}{8}$	$\frac{1}{2}$

Table 7.2: Order of the zero of a form defined by (7.17-7.23) at the cusps of $\widetilde{\Gamma_0(12)}$

Cusp	Zero Order
$1=0$	$A_2/12$
$1/2$	$(15-12A_1+28A_2-8A_3-12A_4-4A_5)/12$
$1/3$	$(-5+4A_1-9A_2+3A_3+4A_4+A_5)/4$
$1/4$	$(-4+3A_1-8A_2+2A_3+4A_4+A_5)/3$
$1/6$	$(25-20A_1+36A_2-12A_3-20A_4-4A_5)/4$
$1/12=i\infty$	A_1

Similarly a form defined by (7.17-7.23) that has a pole of order k at $i\infty$, but no pole at any other cusp can be found by solving the inequalities (7.28 - 7.32) and $-k = A_1$ over $\mathbb{Z}$. For a simple pole at $i\infty$, there are five such Dedekind-η products. They are

$$\psi_1 = \frac{\eta_2^{12}\eta_3}{\eta_1^5\eta_4^4\eta_6\eta_{12}^2} = \frac{1}{\mathbf{q}} + 5 + O[\mathbf{q}], \tag{7.36}$$

$$\frac{\eta_2^3\eta_4^2\eta_6^2}{\eta_1^2\eta_{12}^4} = \frac{1}{\mathbf{q}} + 2 + O[\mathbf{q}], \tag{7.37}$$

$$\frac{\eta_2^2\eta_6^9}{\eta_1\eta_3^3\eta_{12}^6} = \frac{1}{\mathbf{q}} + 1 + O[\mathbf{q}], \tag{7.38}$$

$$\frac{\eta_2^5\eta_3^3}{\eta_1^3\eta_4\eta_{12}^3} = \frac{1}{\mathbf{q}} + 3 + O[\mathbf{q}], \tag{7.39}$$

$$\frac{\eta_1\eta_2^3\eta_6^2}{\eta_3\eta_4\eta_{12}^3} = \frac{1}{\mathbf{q}} - 1 + O[\mathbf{q}]. \tag{7.40}$$

For a triple pole, there are 35 such forms. One of them is

$$\psi_3 = \frac{\eta_2\eta_3^2\eta_4^4\eta_6^4}{\eta_{12}^{10}} = \frac{1}{\mathbf{q}^3} - \frac{1}{\mathbf{q}} - 2 + O[\mathbf{q}]. \tag{7.41}$$

Thus the linear combination

$$f_6 = -6\psi_3 - 2\psi_1 - 2\psi_0 = -\frac{6}{\mathbf{q}^3} + \frac{4}{\mathbf{q}} + O[\mathbf{q}] \tag{7.42}$$

is a vectorizable modular form over $\widetilde{\Gamma_0(12)}$ for $\widetilde{\Gamma_0(12)}$ of weight $\frac{1}{2}$ of character $\chi_\theta\chi_{144}$ with no poles at finite cusps.

Theorem 7.1.5. *There exists a nonzero constant c_6 such that*

$$t_6 = c_6\Psi(F_{f_6})^2. \tag{7.43}$$

Proof. There is an equality of divisors

$$\mathrm{div}(t_6) = 4Z(1, 0; \Gamma^*) - 6Z(3, 0; \Gamma^*) = \mathrm{div}(\Psi(F_{f_6})^2).$$

□

7.1.2 $\Delta = -24$

In this section we calculate $\Psi(F_{f_6})(P_2)$. Recall that P_2 is the CM point with discriminant -24 on the Shimura curve $\mathcal{X}_6^*$. The result of the calculation gives the value of c_6 in Theorem 7.1.5 since by definition $t_6(P_2) = 1$.

Recall that $B = \left(\frac{5,6}{\mathbb{Q}}\right)$. Set $m = 36$ so that by (6.2),

$$L = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3 \tag{7.44}$$

where

$$\ell_1 = \alpha, \qquad \ell_2 = \tfrac{36\alpha+\alpha\beta}{5}, \qquad \ell_3 = \tfrac{\beta+\alpha\beta}{2}. \tag{7.45}$$

Take $z = \ell_3$ so that $Q(z) = 6$. Then the negative plane is spanned by

$$u_1 = 120\ell_2 - 60\ell_3, \tag{7.46}$$

$$u_2 = 62280\ell_1 - 8640\ell_2 + 4320\ell_3, \tag{7.47}$$

and

$$Q(Xu_1 + Yu_2) = -3736800(X^2 + 6Y^2). \tag{7.48}$$

By Lemma 6.1.3, a basis of L_- is given by

$$\ell_1^- = 2\ell_2 - \ell_3, \tag{7.49}$$

$$\ell_2^- = \ell_1, \tag{7.50}$$

and a calculation shows that

$$Q(X\ell_1^- + Y\ell_2^-) = -1038X^2 - 144XY - 5Y^2. \tag{7.51}$$

Thus globally

$$S = \begin{pmatrix} -1038 & -72 \\ \\ -72 & -5 \end{pmatrix}. \tag{7.52}$$

For primes $p \neq 173$, this is $\mathbb{Z}_p$ equivalent to

$$S_p \sim \begin{pmatrix} -173 \cdot 6 & 0 \\ \\ 0 & -173 \end{pmatrix} \tag{7.53}$$

where the basis of $L_{-,p}$ is given by

$$\ell^-_{1,p} = \ell^-_1, \tag{7.54}$$

$$\ell^-_{2,p} = -12\ell^-_1 + 173\ell^-_2. \tag{7.55}$$

When $p = 173$, S is $\mathbb{Z}_{173}$ equivalent to

$$S_{173} \sim \begin{pmatrix} -5 & 0 \\ \\ 0 & -5 \cdot 6 \end{pmatrix} \tag{7.56}$$

where the basis of $L_{-,173}$ is given by

$$\ell^-_{1,173} = \ell^-_2, \tag{7.57}$$

$$\ell^-_{2,173} = 5\ell^-_1 - 72\ell^-_2. \tag{7.58}$$

By Theorem 6.1.2 and its corollary, $L/(L_- + L_+)$ is cyclic of order 2 and $\lambda = \ell_2 + (L_- + L_+)$ represents its nontrivial member. This has the decomposition

$$\lambda_+ = \frac{1}{2}z + L_+, \tag{7.59}$$

$$\lambda_- = \frac{1}{2}\ell^-_1 + L_-. \tag{7.60}$$

By Theorem 4.5.1,

$$\sum_{z \in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-6}))} \log ||\Psi(z, F_{f_6})||^2 = \left(\frac{-1}{4}\right)(-6\kappa_0(3) + 4\kappa_0(1)). \tag{7.61}$$

Considering (4.26),

$$\kappa_0(1) = \kappa_0^-(1), \tag{7.62}$$

$$\kappa_0(3) = \kappa_0^-(3) + \kappa_{\lambda_-}^-(3/2) + \kappa_{\lambda_-}^-(3/2). \tag{7.63}$$

The term $\kappa_{\lambda_-}^-(3/2)$ appears twice in (7.63) due to the two values $x = \pm z/2 \in \lambda_+ + L_+ = (\frac{1}{2} + \mathbb{Z})z$ that satisfy $3 - Q(x) \geq 0$.

Let $\Sigma = \{\text{primes } q \mid \exists p \text{ with } q \mid \det(S_p)\}$. By Corollary 6.2.5, for primes $p \notin \Sigma$ and $\mathrm{ord}_p(m) = 0$, the Whittaker polynomial $W_{m,p}^*(s, u)$ is identically 1. The results at other primes are organized in Tables 7.3 and 7.4 and

$$\kappa_0(1) = -6\log(2), \tag{7.64}$$

$$\kappa_0(3) = -8\log(2) - 4\log(3). \tag{7.65}$$

Thus

$$\sum_{z \in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-6}))} \log ||\Psi(z, F_{f_6})||^2 = -6\log(3) - 6\log(2). \tag{7.66}$$

By [5], the CM point with discriminant -24 is a rational point and

$$\Psi(P_2, F_{f_6})^2 = 6^{-6}. \tag{7.67}$$

Corollary 7.1.6. $t_6 = 6^6 \Psi(F_{f_6})^2$.

Table 7.3: Whittaker Polynomials for $\kappa_0(1)$ when $D = 6$, $\Delta = -24$

u	m	p	u_p	$W^*_{m,p}(s,u)$	$W^*_{m,p}(0,u)$	$W^{*,\prime}_{m,p}(0,u)$	$\kappa^-_u(m)$
0	1	2	0	$\frac{1-X^3}{2\sqrt{2}}$	0	$\frac{3\log(2)}{2\sqrt{2}}$	
		3	0	$\frac{1+X}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$		
		5	0	1	1		
		173	0	1	1		
0	1						$-6\log(2)$

Table 7.4: Whittaker Polynomials for $\kappa_0(3)$ when $D = 6$, $\Delta = -24$

u	m	p	u_p	$W^*_{m,p}(s,u)$	$W^*_{m,p}(0,u)$	$W^{*,\prime}_{m,p}(0,u)$	$\kappa^-_u(m)$
0	3	2	0	$\frac{1+X^3}{2\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		
		3	0	$\frac{1-X^2}{\sqrt{3}}$	0	$\frac{2\log(3)}{\sqrt{3}}$	
		5	0	1	1		
		173	0	1	1		
0	3						$-4\log(3)$
λ_-	3/2	2	$\frac{-\ell^-_{1,p}}{2}$	$\frac{1-X^2}{2\sqrt{2}}$	0	$\frac{\log(2)}{\sqrt{2}}$	
		3	$\frac{-\ell^-_{1,p}}{2}$	$\frac{1+X^2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$		
		5	$\frac{-\ell^-_{1,p}}{2}$	1	1		
		173	$\frac{-36\ell^-_{1,p}}{5} - \frac{\ell^-_{2,p}}{10}$	1	1		
λ_-	3/2						$-4\log(2)$

7.1.3 $\Delta = -163$

We are now able to compute the coordinates of the other rational CM points listed in Table 8.1. We illustrate the calculations with the example of $\Delta = -163$. The calculations of the Whittaker polynomials was implemented in Mathematica and are omitted here. As an example, the Mathematica code and output for $\Delta = -52$ and $D = 10$ can be found in the Appendix.

Take $z = 85\ell_1 - 12\ell_2$ so that $Q(z) = 163$. Then the negative plane is spanned by

$$u_1 = -720\ell_2 - 720\ell_3, \tag{7.68}$$

$$u_2 = 2125440\ell_1 - 295920\ell_2 + 117360\ell_3, \tag{7.69}$$

and

$$Q(Xu_1 + Yu_2) = -127526400(X^2 + 6Y^2). \tag{7.70}$$

By Lemma 6.1.3, a basis of L_- is given by

$$\ell_1^- = -\ell_2 - \ell_3, \tag{7.71}$$

$$\ell_2^- = 36\ell_1 - 7\ell_2, \tag{7.72}$$

and

$$Q(X\ell_1^- + Y\ell_2^-) = -6(41X^2 + 163XY + 163Y^2). \tag{7.73}$$

Thus globally

$$S = \begin{pmatrix} -246 & -489 \\ -489 & -978 \end{pmatrix}. \tag{7.74}$$

Table 7.5: $\mathbb{Z}_p$ equivalences for S when $D = 6$, $\Delta = -163$

p	S_p	$\ell_{1,p}^-$	$\ell_{2,p}^-$
$p \neq 2, 41$	$\begin{pmatrix} -246 & 0 \\ 0 & -40098 \end{pmatrix}$	ℓ_1^-	$-163\ell_1^- + 82\ell_2^-$
2	$-246\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$	ℓ_1^-	$\frac{163(1+\sqrt{489})}{326}\ell_1^- - 41\sqrt{\frac{3}{163}}\ell_2^-$
41	$\begin{pmatrix} -978 & 0 \\ 0 & -6 \end{pmatrix}$	ℓ_2^-	$2\ell_1^- - \ell_2^-$

This has the $\mathbb{Z}_p$ equivalences given in Table (7.5).

By Theorem 6.1.2 and its corollary, $L/(L_- + L_+)$ is cyclic of order 163 and $\lambda = \ell_3 + (L_- + L_+)$ represents a generator. This has the decomposition

$$\lambda_+ = \frac{-36}{163}z + L_+, \tag{7.75}$$

$$\lambda_- = -\ell_1^- + \frac{85}{163}\ell_2^- + L_-. \tag{7.76}$$

Then computations of Whittaker polynomials as before yield

$$\kappa_0(1) = -4\log(2) - 11\log(3) - 4\log(7) - 4\log(19) - 4\log(23), \tag{7.77}$$

$$\kappa_0(3) = -\frac{40}{3}\log(2) - 4\log(3) - 4\log(5) - 4\log(11) - 4\log(17). \tag{7.78}$$

Thus by Theorem 4.5.1 the CM point $\mathcal{P}_{-163}$ with discriminant -163 has

$$t_6(\mathcal{P}_{-163}) = \frac{3^{11}7^4 19^4 23^4}{2^{10}5^6 11^6 17^6}. \tag{7.79}$$

Note that this proves the conjectural value given in Table 2 of [5].

7.2 $D = 10$

7.2.1 The Input Form

Now we consider the quaternion algebra ramified at the primes 2 and 5. In [5] it given as $\mathbb{Q} + \mathbb{Q}b + \mathbb{Q}e + \mathbb{Q}be$ with

$$b^2 + 2 = e^2 - 5 = be + eb = 0. \tag{7.80}$$

Again, there is an isomorphism $\mathbb{Q} + \mathbb{Q}b + \mathbb{Q}e + \mathbb{Q}be \xrightarrow{\sim} B = \left(\frac{13,10}{\mathbb{Q}}\right)$ given by

$$b \mapsto -\frac{8}{13}\alpha - \frac{3}{13}\alpha\beta, \tag{7.81}$$

$$e \mapsto \frac{15}{13}\alpha + \frac{4}{13}\alpha\beta, \tag{7.82}$$

$$be \mapsto \beta. \tag{7.83}$$

B contains the maximal order

$$\mathcal{O} = \mathbb{Z} + \left(\frac{-8\alpha - 3\alpha\beta}{13}\right)\mathbb{Z} + \left(\frac{13 + 15\alpha + 4\alpha\beta}{26}\right)\mathbb{Z} + \left(\frac{-8\alpha + 13\beta - 3\alpha\beta}{26}\right)\mathbb{Z}.$$

Then by Section 4.1 of [5], the image of $\Gamma^* \subset \mathrm{PGL}_2(\mathbb{R})$ is presented as

$$< s_2, s_2', s_2'', s_3 \mid s_2^2 = s_2'^2 = s_2''^2 = s_3^3 = s_2 s_2'' s_2' s_3 = 1 >, \tag{7.84}$$

with

$$s_2 = -\frac{8}{13}\alpha - \frac{3}{13}\alpha\beta, \tag{7.85}$$

$$s_2' = -\frac{35}{13}\alpha - \frac{1}{2}\beta - \frac{23}{26}\alpha\beta, \tag{7.86}$$

$$s_2'' = -\frac{20}{13}\alpha - \frac{1}{2}\beta - \frac{15}{26}\alpha\beta, \tag{7.87}$$

$$s_3 = -\frac{1}{2} - \frac{31}{26}\alpha - \frac{5}{13}\alpha\beta. \tag{7.88}$$

There is a map $t_{10} : \mathcal{X}_{10}^* \xrightarrow{\sim} \mathbb{P}^1$ such that

$$\begin{aligned} \mathrm{div}(t_{10}) &= P_3 - P_2, \\ t_{10}(P_2') &= 2, \end{aligned} \tag{7.89}$$

where P_2, P_2', P_3 are the fixed points of s_2, s_2', s_3, respectively. Again the fixed point of s_i is the fixed point of all of $\boldsymbol{k}_i^\times \subset B^\times$ where $\boldsymbol{k}_i = \mathbb{Q}(s_i^0)$. Now

$$\boldsymbol{k}_2 \simeq \mathbb{Q}(\sqrt{-2}), \qquad \boldsymbol{k}_2' \simeq \mathbb{Q}(\sqrt{-5}), \qquad \boldsymbol{k}_2'' \simeq \mathbb{Q}(\sqrt{-10}), \qquad \boldsymbol{k}_3 \simeq \mathbb{Q}(\sqrt{-3}). \tag{7.90}$$

Lemma 7.2.1. *The stabilizers of the s_i^0 have the following sizes.*

1) $|Stab_{\Gamma^*}(s_2^0)| = 2$.

2) $|Stab_{\Gamma^*}(s_3^0)| = 3$.

Proof. These values are obtained in the same manner as in Lemma 7.1.1. □

Lemma 7.2.2. *The following equalities hold.*

1) $Z(2, 0; \Gamma^*) = \frac{1}{2} P_2$.

2) $Z(3, 0; \Gamma^*) = \frac{1}{3} P_3$.

Proposition 7.2.3. *The following identity for t_{10} holds,*

$$div(t_{10}) \quad = \quad 3Z(3, \Theta_0; \Gamma^*) - 2Z(2, \Theta_0; \Gamma^*). \tag{7.91}$$

Then the same line of reasoning as in Section 7.1.1 applied to the case $|\Lambda_L| = 200$ and $N = 20$ gives the following result.

Theorem 7.2.4. *Let*

$$f_{10} = 3\left(\frac{\eta_4^6\eta_{10}^8}{\eta_2^3\eta_5^2\eta_{20}^8}\right) - 2\left(\frac{\eta_2^3\eta_4^2\eta_{10}^2}{\eta_1^2\eta_{20}^4}\right) - 5\left(\frac{\eta_4^2\eta_{10}^6}{\eta_2\eta_5^2\eta_{20}^4}\right) + 4\left(\frac{\eta_2^5}{\eta_1^2\eta_4^2}\right) \quad (7.92)$$

$$= \frac{3}{\mathbf{q}^3} - \frac{2}{\mathbf{q}^2} + O[\mathbf{q}]. \quad (7.93)$$

It is a vectorizable modular form over $\widetilde{\Gamma_0(20)}$ with no poles at finite cusps. Thus

$$div(t_{10}) = div(\Psi(F_{f_{10}})^2). \quad (7.94)$$

So again the two functions agree up to a nonzero constant,

$$t_{10} = c_{10}\Psi(F_{f_{10}})^2. \quad (7.95)$$

7.2.2 $\Delta = -20$

To compute the constant c_{10}, we now consider the case of $\Delta = -20$. Recall that P_2' is the CM point with discriminant -20 on the Shimura curve $\mathcal{X}_{10}^*$ and $t_{10}(P_2') = 2$ by definition. Set $B = \left(\frac{13,10}{\mathbb{Q}}\right)$ and $m = 240$ so that by (6.2),

$$L = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3 \quad (7.96)$$

where

$$\ell_1 = \alpha, \qquad \ell_2 = \tfrac{240\alpha+\alpha\beta}{13}, \qquad \ell_3 = \tfrac{\beta+\alpha\beta}{2}. \quad (7.97)$$

Take $z = 55\ell_1 - 3\ell_2$ so that $Q(z) = 5$. Then the negative plane is spanned by

$$u_1 = -780\ell_2 - 4680\ell_3, \quad (7.98)$$

$$u_2 = 7698600\ell_1 - 417300\ell_2 + 62400\ell_3, \quad (7.99)$$

Table 7.6: $\mathbb{Z}_p$ equivalences for S when $D = 10$, $\Delta = -20$

p	S_p	$\ell^-_{1,p}$	$\ell^-_{2,p}$
$p \neq 2, 7, 47$	$\begin{pmatrix} -3290 & 0 \\ 0 & -658 \end{pmatrix}$	ℓ_1^-	$107\ell_1^- + 658\ell_2^-$
$p = 2, 7, 47$	$\begin{pmatrix} -87 & 0 \\ 0 & -435 \end{pmatrix}$	ℓ_2^-	$87\ell_1^- + 535\ell_2^-$

and

$$Q(Xu_1 + Yu_2) = -2001636000(X^2 + 5Y^2). \tag{7.100}$$

By Lemma 6.1.3, a basis of L_- is given by

$$\ell_1^- = -\ell_2 - 6\ell_3, \tag{7.101}$$

$$\ell_2^- = 3\ell_1 + \ell_3, \tag{7.102}$$

and

$$Q(X\ell_1^- + Y\ell_2^-) = -3290X^2 + 1070XY - 87Y^2. \tag{7.103}$$

Thus globally

$$S = \begin{pmatrix} -3290 & 535 \\ 535 & -87 \end{pmatrix}. \tag{7.104}$$

This has the $\mathbb{Z}_p$ equivalences given in Table 7.6.

By Theorem 6.1.2 and its corollary, $L/(L_- + L_+)$ is trivial. Theorem 4.5.1

yields,

$$\sum_{z \in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-5}))} \log ||\Psi(z, F_{f_{10}})||^2 = \left(\frac{-1}{4}\right)(3\kappa_0(3) - 2\kappa_0(2)). \qquad (7.105)$$

Computations as before yield

$$\kappa_0(2) = -6\log(2), \qquad (7.106)$$

$$\kappa_0(3) = -8\log(2). \qquad (7.107)$$

Thus

$$\sum_{z \in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-5}))} \log ||\Psi(z, F_{f_{10}})||^2 = 3\log(2). \qquad (7.108)$$

By [5], the CM point with discriminant -20 is a rational point and

$$\Psi(P_2', F_{f_{10}})^2 = 2^3. \qquad (7.109)$$

Since $t_{10}(P_2') = 2$,

$$t_{10} = 2^{-2}\Psi(F_{f_{10}})^2. \qquad (7.110)$$

7.2.3 $\Delta = -68$

Again, we are now able to compute the coordinates of the other rational CM points for $\mathcal{X}_{10}^*$ listed in Table 8.3. Moreover, we are also capable of calculating the norms of irrational CM points. As an example, we compute the norm of the irrational CM point with discriminant -68.

Take $z = 221\ell_1 - 12\ell_2 + \ell_3$ so that $Q(z) = 17$. Then the negative plane is spanned by

$$u_1 = -1560\ell_2 - 6500\ell_3, \qquad (7.111)$$

$$u_2 = 36199800\ell_1 - 1962480\ell_2 + 269620\ell_3, \qquad (7.112)$$

Table 7.7: $\mathbb{Z}_p$ equivalences for S when $D = 10$, $\Delta = -68$

p	S_p	$\ell^-_{1,p}$	$\ell^-_{2,p}$
$p \neq 35023$	$\begin{pmatrix} -175115 & 0 \\ 0 & -17 \cdot 175115 \end{pmatrix}$	ℓ_2^-	$35023\ell_1^- - 31229\ell_2^-$
35023	$\begin{pmatrix} -17 \cdot 8190 & 0 \\ 0 & 8190 \end{pmatrix}$	ℓ_1^-	$-1837\ell_1^- + 1638\ell_2^-$

and

$$Q(Xu_1 + Yu_2) = -9411948000(X^2 + 17Y^2). \tag{7.113}$$

By Lemma 6.1.3, a basis of L_- is given by

$$\ell_1^- = -6\ell_2 - 25\ell_3, \tag{7.114}$$

$$\ell_2^- = 5\ell_1 - 7\ell_2 - 28\ell_3, \tag{7.115}$$

and

$$Q(X\ell_1^- + Y\ell_2^-) = -5(27846X^2 + 62458XY + 35023Y^2). \tag{7.116}$$

Then S has the $\mathbb{Z}_p$ equivalences given in Table 7.7.

By Theorem 6.1.2 and its corollary, $L/(L_- + L_+)$ is cyclic of order 17 and is generated by $\lambda = \ell_3 + (L_- + L_+)$. This has the decomposition

$$\lambda_+ = \frac{-30}{17} z + L_+, \tag{7.117}$$

$$\lambda_- = \frac{-1487}{17} \ell_1^- + 78\ell_2^- + L_-. \tag{7.118}$$

Then computations as before yield

$$\kappa_0(1) = -6\log(2) - 6\log(5), \tag{7.119}$$

$$\kappa_0(3) = -8\log(2) - \frac{14}{3}\log(5). \tag{7.120}$$

This time the CM point with discriminant -68 is irrational, and Theorem 4.5.1 gives its norm (after renormalization) as $2^2 \cdot 5$.

Chapter 8

Tables

8.1 $D = 6$

8.1.1 Coordinates of Rational CM Points on $\mathcal{X}_6^*$

The following table gives the values of t_6 (as defined by (7.10)) at the rational CM points of $\mathcal{X}_6^*$. These values verify the values in Table 2 of [5]. Denote $t_6(P_{CM}) = (r : s)$.

Table 8.1: Coordinates of Rational CM Points on $\mathcal{X}_6^*$

Δ	r	s	Proved in [5]
-3	1	0	Y
-4	0	1	Y
-24	1	1	Y
-40	3^7	5^3	Y
-52	$2^2 3^7$	5^6	Y
-19	3^7	2^{10}	Y
-84	$-2^2 7^2$	3^3	Y
-88	$3^7 7^4$	$5^6 11^3$	Y

Table 8.1: Coordinates of Rational CM Points on $\mathcal{X}_6^*$

Δ	r	s	Proved in [5]
-100	$2^4 3^7 7^4 5$	11^6	Y
-120	7^4	$3^3 5^3$	Y
-132	$2^4 11^2$	5^6	Y
-148	$2^2 3^7 7^4 11^4$	$5^6 17^6$	N
-168	$-7^2 11^4$	5^6	Y
-43	$3^7 7^4$	$2^{10} 5^6$	Y
-51	-7^4	2^{10}	Y
-228	$2^6 7^4 19^2$	$3^6 5^6$	N
-232	$3^7 7^4 11^4 19^4$	$5^6 23^6 29^3$	N
-67	$3^7 7^4 11^4$	$2^{16} 5^6$	N
-75	11^4	$2^{10} 3^3 5$	Y
-312	$7^4 23^4$	$5^6 11^6$	Y
-372	$-2^2 7^4 19^4 31^2$	$3^3 5^6 11^6$	N
-408	$-7^4 11^4 31^4$	$3^6 5^6 17^3$	N
-123	$-7^4 19^4$	$2^{10} 5^6$	N
-147	$-11^4 23^4$	$2^{10} 3^3 5^6 7$	Y
-163	$3^{11} 7^4 19^4 23^4$	$2^{10} 5^6 11^6 17^6$	N
-708	$2^8 7^4 11^4 47^4 59^2$	$5^6 17^6 29^6$	N
-267	$-7^4 31^4 43^4$	$2^{16} 5^6 11^6$	N

8.1.2 Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Here we give the norms for all CM points of fundamental discriminant $\Delta = d$ or $4d$ for $0 < -d \leq 250$. This cut-off is arbitrary. It is also only for implementation reasons that we only compute for fundamental discriminants (i.e. d squarefree).

Table 8.2: Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-40	$\frac{3^7}{5^3}$	$\frac{2^3 17^2}{5^3}$
-52	$\frac{2^2 3^7}{5^6}$	$\frac{13^1 23^2}{5^6}$
-19	$\frac{3^7}{2^{10}}$	$\frac{13^2 19^1}{2^{10}}$
-84	$\frac{2^2 7^2}{3^3}$	$\frac{13^2}{3^3}$
-88	$\frac{3^7 7^4}{5^6 11^3}$	$\frac{2^5 17^2 41^2}{5^6 11^3}$
-120	$\frac{7^4}{3^3 5^3}$	$\frac{2^4 19^2}{3^3 5^3}$
-132	$\frac{2^4 11^2}{5^6}$	$\frac{3^4 13^2}{5^6}$
-136	$\frac{3^{14}}{11^2 17^3}$	$\frac{2^6 13^4 41^2}{11^6 17^2}$
-148	$\frac{2^2 3^7 7^4 11^4}{5^6 17^6}$	$\frac{13^2 37^1 47^2 71^2}{5^6 17^6}$
-168	$\frac{7^2 11^4}{5^6}$	$\frac{2^3 3^5 19^2}{5^6}$
-43	$\frac{3^7 7^4}{2^{10} 5^6}$	$\frac{19^2 37^2 43^1}{2^{10} 5^6}$
-184	$\frac{3^{14} 7^8}{17^6 23^3}$	$\frac{2^8 13^4 89^2}{17^4 23^2}$

Table 8.2: Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-51	$\frac{7^4}{2^{10}}$	$\frac{3^4 17^1}{2^{10}}$
-228	$\frac{2^6 7^4 19^2}{3^6 5^6}$	$\frac{13^2 17^2 37^2}{3^6 5^6}$
-232	$\frac{3^7 7^4 11^4 19^4}{5^6 23^6 29^3}$	$\frac{2^3 13^2 17^2 41^2 89^2 113^2}{5^6 23^6 29^3}$
-244	$\frac{2^6 3^{21} 19^4}{17^6 29^6}$	$\frac{19^4 37^2 47^2 61^1}{17^2 29^6}$
-264	$\frac{19^4}{3^9 11^1}$	$\frac{2^6 19^2 43^2}{3^9 11^3}$
-67	$\frac{3^7 7^4 11^4}{2^{16} 5^6}$	$\frac{13^2 43^2 61^2 67^1}{2^{16} 5^6}$
-276	$\frac{2^4 23^2}{11^2}$	$\frac{3^8 23^1 37^2}{11^6}$
-280	$\frac{3^{14} 7^4 23^4}{5^6 11^2 29^6}$	$\frac{2^{12} 13^4 23^2 113^2 137^2}{5^6 11^6 29^6}$
-292	$\frac{2^{10} 3^{14} 19^4}{5^{12} 23^2}$	$\frac{13^4 17^4 19^2 67^2 71^2}{5^{12} 23^6}$
-312	$\frac{7^4 23^4}{5^6 11^6}$	$\frac{2^4 3^5 13^1 17^2 43^2}{5^6 11^6}$
-328	$\frac{3^{18} 11^8 19^4}{5^{12} 17^6 41^3}$	$\frac{2^6 19^2 23^4 89^2 137^2}{5^{12} 17^4 41^2}$
-340	$\frac{2^4 3^{18} 7^8 23^4}{5^6 29^6 41^6}$	$\frac{13^4 17^2 19^4 23^2 61^2 167^2}{5^6 29^6 41^6}$
-91	$\frac{3^{14} 7^4}{2^{26} 11^2}$	$\frac{13^2 17^4 37^2 67^2}{2^{26} 11^6}$
-372	$\frac{2^2 7^4 19^4 31^2}{3^3 5^6 11^6}$	$\frac{13^2 23^2 37^2 61^2}{3^3 5^6 11^6}$
-376	$\frac{3^{28} 31^4}{23^2 41^6 47^3}$	$\frac{2^{16} 37^4 113^2}{23^2 41^4 47^2}$
-388	$\frac{2^{14} 3^{18} 31^4}{5^{12} 11^2 47^6}$	$\frac{13^4 17^4 43^2 167^2 191^2}{5^{12} 11^6 47^4}$
-408	$\frac{7^4 11^4 31^4}{3^6 5^6 17^3}$	$\frac{2^6 13^2 19^2 43^2 67^2}{3^6 5^6 17^3}$
-420	$\frac{2^{12} 7^4 23^4}{5^6 17^6}$	$\frac{3^8 23^2 61^2}{5^6 17^4}$

Table 8.2: Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-424	$\frac{3^{25}7^{12}19^4}{29^6 47^6 53^3}$	$\frac{2^9 13^6 19^4 37^4 41^2 137^2}{29^6 47^6 53^3}$
-436	$\frac{2^6 3^{21} 7^{12} 31^4}{17^6 41^6 53^6}$	$\frac{13^6 43^4 71^2 109^1 191^2}{17^2 41^6 53^6}$
-456	$\frac{7^8 19^2}{11^2 17^6}$	$\frac{2^6 3^9 19^1 67^2}{11^6 17^4}$
-115	$\frac{3^{14} 19^4}{2^{20} 5^6 11^2}$	$\frac{13^4 19^2 23^2 61^2 109^2}{2^{20} 5^6 11^6}$
-472	$\frac{3^{21} 19^4 23^4 31^4}{5^{18} 53^6 59^3}$	$\frac{2^{17} 19^4 23^4 47^2 89^2 233^2}{5^{18} 53^6 59^3}$
-123	$\frac{7^4 19^4}{2^{10} 5^6}$	$\frac{3^4 13^2 23^2 41^1}{2^{10} 5^6}$
-516	$\frac{2^{14} 31^4 43^2}{3^{12} 17^6}$	$\frac{37^2 41^2 61^2}{3^{12} 17^2}$
-520	$\frac{3^{18} 7^8 19^4 43^4}{5^6 11^2 41^6 59^6}$	$\frac{2^6 13^2 17^4 19^2 37^4 113^2 233^2 257^2}{5^6 11^6 41^6 59^6}$
-532	$\frac{2^4 3^{14} 7^4 11^8 23^4 43^4}{5^{12} 29^6 53^6}$	$\frac{17^4 19^2 23^2 37^2 109^2 191^2 239^2 263^2}{5^{12} 29^6 53^6}$
-552	$\frac{19^4 43^4}{3^6 5^{12} 23^1}$	$\frac{2^6 13^4 19^2 43^2 67^2}{3^6 5^{12} 23^3}$
-139	$\frac{3^{21} 19^4 23^4}{2^{36} 17^6}$	$\frac{19^4 23^4 43^2 139^1}{2^{36} 17^2}$
-564	$\frac{2^4 7^8 47^2}{11^2 23^6}$	$\frac{3^8 13^4 17^4 47^1}{11^6 23^4}$
-568	$\frac{3^{14} 7^8 23^4 31^4}{5^{12} 17^6 47^2 71^3}$	$\frac{2^8 19^4 23^2 41^2 137^2 257^2 281^2}{5^{12} 17^4 47^6 71^2}$
-580	$\frac{2^{20} 3^{32} 43^4 47^4}{5^{12} 59^6 71^6}$	$\frac{13^8 41^4 43^4 47^2 139^2 263^2}{5^{12} 59^6 71^6}$
-616	$\frac{3^{28} 7^8 43^4}{11^2 23^2 53^6 71^6}$	$\frac{2^{12} 13^8 37^2 61^4 233^2 281^2}{11^6 23^2 53^6 71^6}$
-628	$\frac{2^6 3^{21} 19^4 31^4 47^4}{5^{18} 11^8 41^6}$	$\frac{19^4 61^2 71^2 157^1 167^2 239^2 311^2}{5^{18} 11^{12} 41^2}$
-163	$\frac{3^{11} 7^4 19^4 23^4}{2^{10} 5^6 11^6 17^6}$	$\frac{13^2 67^2 109^2 139^2 157^2 163^1}{2^{10} 5^6 11^6 17^6}$
-660	$\frac{2^4 7^8 11^4 43^4}{3^{12} 5^6 23^6}$	$\frac{17^4 47^2 61^2 109^2}{3^{12} 5^6 23^4}$

Table 8.2: Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert (1-t_6)(\mathcal{P}_\Delta)\rvert$
-664	$\frac{3^{39}47^4}{29^6 59^6 83^3}$	$\frac{2^{27}37^4 47^4 61^4 71^2 89^2 257^2}{11^{12}29^6 59^6 83^3}$
-696	$\frac{11^2 23^4 31^4}{3^{15}29^3}$	$\frac{2^{12}13^6 23^4 67^2}{3^{15}11^6 29^3}$
-708	$\frac{2^8 7^4 11^4 47^4 59^2}{5^6 17^6 29^6}$	$\frac{3^4 13^2 19^2 23^2 37^2 41^2 109^2}{5^6 17^6 29^6}$
-712	$\frac{3^{32}19^8 43^4 59^4}{5^{24}83^6 89^3}$	$\frac{2^{12}19^4 41^2 43^2 47^4 113^2 281^2 353^2}{5^{24}83^6 89^2}$
-724	$\frac{2^{10}3^{39}59^4}{17^6 53^6 89^6}$	$\frac{17^2 41^4 43^4 67^4 157^2 181^1 359^2}{11^{12}53^6 89^6}$
-744	$\frac{7^{12}31^2 59^4}{23^6 29^6}$	$\frac{2^9 3^{13}41^2 43^2}{23^2 29^6}$
-187	$\frac{3^{18}11^4 31^4}{2^{20}5^{12}23^2}$	$\frac{13^4 17^2 19^4 37^2 163^2 181^2}{2^{20}5^{12}23^6}$
-760	$\frac{3^{14}7^8 11^8 23^4 31^4 47^4}{5^6 41^6 71^6 89^6}$	$\frac{2^8 13^4 17^4 19^2 23^2 47^2 61^4 137^2 233^2 353^2}{5^6 41^6 71^6 89^6}$
-772	$\frac{2^{14}3^{14}7^8 31^4 43^4}{5^{12}23^6 59^2 83^6}$	$\frac{13^4 17^4 43^2 139^2 239^2 311^2 359^2 383^2}{5^{12}23^4 59^6 83^6}$
-195	$\frac{19^4 31^4}{2^{26}5^6}$	$\frac{3^8 13^2 19^2 47^2}{2^{26}5^6}$
-804	$\frac{2^{18}11^2 19^4 67^2}{3^{12}29^6}$	$\frac{13^6 17^6 19^4 109^2}{3^{12}11^6 29^6}$
-808	$\frac{3^{25}23^4 31^4 59^4 67^4}{5^{18}11^8 47^6 101^3}$	$\frac{2^9 13^6 23^4 37^2 41^2 89^2 257^2 401^2}{5^{18}11^{12}47^2 101^3}$
-820	$\frac{2^8 3^{28}7^{16}47^4 67^4}{5^{12}29^6 89^6 101^6}$	$\frac{37^4 41^2 47^2 67^4 109^2 167^2 181^2 263^2 383^2}{5^{12}29^6 89^6 101^6}$
-840	$\frac{7^4 43^4 67^4}{3^{12}5^6 11^2 17^6}$	$\frac{2^6 13^4 19^4 23^4 139^2}{3^{12}5^6 11^6 17^4}$
-211	$\frac{3^{25}7^{12}31^4}{2^{36}17^6 23^6}$	$\frac{41^4 61^2 157^2 211^1}{2^{36}17^2 23^2}$
-852	$\frac{2^4 59^4 71^2}{5^{12}11^2 23^2}$	$\frac{3^8 13^4 19^4 47^2 61^2 71^1}{5^{12}11^6 23^6}$
-856	$\frac{3^{21}7^{12}11^2 19^4 31^4 71^4}{53^6 83^6 101^6 107^3}$	$\frac{2^{19}13^6 17^6 19^4 37^4 281^2 353^2 401^2}{11^6 53^6 83^6 101^6 107^3}$
-868	$\frac{2^{28}3^{28}7^8 67^4}{5^{24}71^2 107^6}$	$\frac{37^4 41^4 67^2 163^2 191^2 211^2 359^2 431^2}{5^{24}71^4 107^6}$

Table 8.2: Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-219	$\frac{7^8 23^4}{2^{26} 3^9}$	$\frac{13^4 23^2 41^2 71^2}{2^{26} 3^9}$
-888	$\frac{31^4 47^4 71^4}{5^{18} 29^6}$	$\frac{2^{12} 3^{13} 37^1 41^2 67^2 139^2}{5^{18} 29^6}$
-904	$\frac{3^{32} 7^{16} 19^4 67^4}{17^6 59^2 89^6 107^6 113^3}$	$\frac{2^{12} 13^8 19^6 43^2 61^4 449^2}{59^6 89^4 107^6 113^2}$
-916	$\frac{2^{10} 3^{35} 19^4 43^4 71^4}{41^6 101^6 113^6}$	$\frac{13^{10} 19^8 43^4 229^1 311^2 383^2 431^2}{11^{12} 41^2 101^6 113^6}$
-235	$\frac{3^{14} 7^8 19^4 31^4}{2^{20} 5^6 11^2 29^6}$	$\frac{17^4 19^2 47^2 139^2 181^2 211^2 229^2}{2^{20} 5^6 11^6 29^6}$
-948	$\frac{2^6 19^4 31^4 67^4 79^2}{3^{15} 5^{18}}$	$\frac{19^4 37^2 47^2 71^2 109^2 157^2}{3^{15} 5^{18}}$
-952	$\frac{3^{28} 7^8 23^2 71^4 79^4}{5^{24} 17^6 113^6}$	$\frac{2^{16} 43^4 47^4 71^2 233^2 401^2 449^2}{5^{24} 17^4 23^4 113^4}$
-964	$\frac{2^{34} 3^{42} 59^4 79^4}{17^{12} 47^2 83^6 107^6}$	$\frac{13^{12} 37^4 67^4 239^2 479^2}{17^4 47^2 83^6 107^6}$
-984	$\frac{7^{12} 11^2 79^4}{3^{12} 23^6 41^3}$	$\frac{2^{16} 37^2 43^2 139^2 163^2}{3^{12} 11^6 23^2 41^3}$
-996	$\frac{2^{16} 7^{12} 71^4 83^2}{17^6 29^6 41^6}$	$\frac{3^{14} 13^6 47^2 157^2}{17^2 29^6 41^6}$

8.2 $D = 10$

8.2.1 Coordinates of Rational CM Points on $\mathcal{X}_{10}^*$

The following table gives the values of t_{10} (as defined by (7.89)) at the rational CM points of $\mathcal{X}_{10}^*$. These values verify the values in Table 4 of [5]. Again denote $t_{10}(P_{CM}) = (r : s)$.

Table 8.3: Coordinates of Rational CM Points on $\mathcal{X}_{10}^*$

Δ	r	s	Proved in [5]
-3	0	1	Y
-8	1	0	Y
-20	2	1	Y
-40	3^3	1	Y
-52	$-2^1 3^3$	5^2	N
-72	5^3	$3^1 7^2$	Y
-120	-3^3	7^2	Y
-88	$3^3 5^3$	$2^1 7^2$	N
-27	$-2^6 3$	5^2	Y
-35	2^6	7	Y
-148	$2^1 3^3 11^3$	$5^2 7^2 13^2$	N

Table 8.3: Coordinates of Rational CM Points on $\mathcal{X}_{10}^{*}$

Δ	r	s	Proved in [5]
-43	$2^6 3^3$	$5^2 7^2$	N
-180	$-2^1 11^3$	13^2	Y
-232	$3^3 11^3 17^3$	$2^2 5^2 7^2 23^2$	N
-67	$-2^6 3^3 5^3$	$7^2 13^2$	N
-280	$3^3 11^3$	$2^1 7^1 23^2$	N
-340	$2^1 3^3 23^3$	$7^2 29^2$	N
-115	$2^9 3^3$	$13^2 23$	N
-520	$3^3 29^3$	$2^3 7^2 13^1 47^2$	N
-163	$-2^9 3^3 5^3 11^3$	$7^2 13^2 29^2 31^2$	N
-760	$3^3 17^3 47^3$	$7^2 31^2 71^2$	N
-235	$2^6 3^3 17^3$	$7^2 37^2 47$	N

8.2.2 Norms of CM Points on $\mathcal{X}_{10}^*$ for $0 < -d \leq 250$

Table 8.4: Norms of CM Points on $\mathcal{X}_{10}^*$ for $0 < -d \leq 250$

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert(2-t_{10})(\mathcal{P}_\Delta)\rvert$
-40	$\frac{3^3}{1}$	$\frac{5^2}{1}$
-52	$\frac{2^1 3^3}{5^2}$	$\frac{2^3 13^1}{5^2}$
-68	$\frac{2^2 5^1}{1}$	$\frac{2^4 17^1}{5^2}$
-88	$\frac{3^3 5^3}{2^1 7^2}$	$\frac{11^1 17^2}{2^1 7^2}$
-120	$\frac{3^3}{7^2}$	$\frac{5^3}{7^2}$
-132	$\frac{2^2 3^6 5^1}{13^2}$	$\frac{2^4 11^2}{5^2}$
-35	$\frac{2^6}{7^1}$	$\frac{2^1 5^2}{7^1}$
-148	$\frac{2^1 3^3 11^3}{5^2 7^2 13^2}$	$\frac{2^5 17^2 37^1}{5^2 7^2 13^2}$
-152	$\frac{11^3}{2^1 5^1}$	$\frac{11^4 19^1}{2^1 5^4}$
-168	$\frac{3^6 11^3}{2^2 5^4 7^2}$	$\frac{11^2 37^2}{2^2 5^4 7^2}$
-43	$\frac{2^6 3^3}{5^2 7^2}$	$\frac{2^1 19^2}{5^2 7^2}$
-212	$\frac{2^3 5^4 11^3}{7^6}$	$\frac{2^{11} 11^4 53^1}{5^2 7^6}$
-228	$\frac{2^2 3^6 5^1 17^3}{7^4 13^2}$	$\frac{2^4 19^2 37^2}{5^2 7^4}$
-232	$\frac{3^3 11^3 17^3}{2^2 5^2 7^2 23^2}$	$\frac{13^2 19^2 53^2}{2^2 5^2 7^2 23^2}$
-248	$\frac{5^2 17^3}{2^2 23^2}$	$\frac{17^2 19^4 31^1}{2^2 5^4 23^2}$
-260	$\frac{2^2 17^3}{7^4 13^1}$	$\frac{2^4 5^4}{7^4}$
-67	$\frac{2^6 3^3 5^3}{7^2 13^2}$	$\frac{2^1 11^2 31^2}{7^2 13^2}$

Table 8.4: Norms of CM Points on $\mathcal{X}_{10}^*$ for $0 < -d \leq 250$

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert (2-t_{10})(\mathcal{P}_\Delta)\rvert$
-280	$\frac{3^3 11^3}{2^1 7^1 23^2}$	$\frac{5^3 13^2}{2^1 7^1 23^2}$
-292	$\frac{2^2 3^6 5^1 17^3}{13^4 29^2}$	$\frac{2^4 17^2 53^2 73^1}{5^2 13^4 29^2}$
-308	$\frac{2^4 5^2 11^3 23^3}{7^4 29^2}$	$\frac{2^{14} 11^2 19^4}{5^4 7^4 29^2}$
-312	$\frac{3^6 17^3 23^3}{2^2 5^4 7^4 31^2}$	$\frac{11^4 13^2 73^2}{2^2 5^4 7^4 31^2}$
-328	$\frac{3^6 5^1 23^1}{2^3 31^2}$	$\frac{11^4 37^2}{2^3 5^2 23^2}$
-83	$\frac{2^{18}}{5^1 13^2}$	$\frac{2^3 13^2 19^2}{5^4}$
-340	$\frac{2^1 3^3 23^3}{7^2 29^2}$	$\frac{2^3 5^2 13^2 17^1}{7^2 29^2}$
-372	$\frac{2^2 3^9 11^3 23^3}{5^4 7^4 13^2 37^2}$	$\frac{2^8 11^2 31^2 73^2}{5^4 7^4 37^2}$
-388	$\frac{2^2 3^6 17^3 29^1}{5^4 13^2 37^2}$	$\frac{2^4 11^4 17^2 97^1}{5^4 29^2 37^2}$
-408	$\frac{3^9 5^1 11^3 29^3}{7^4 13^4 31^2}$	$\frac{11^2 17^2 19^4 97^2}{5^2 7^4 13^4 31^2}$
-420	$\frac{2^2 3^6 29^3}{7^2 37^2}$	$\frac{2^4 5^6 17^2}{7^2 37^2}$
-107	$\frac{2^{21}}{5^1 7^6}$	$\frac{2^3 17^4 31^2}{5^4 7^6}$
-440	$\frac{11^3 23^3}{2^2 13^4}$	$\frac{5^6 11^1}{2^2 13^2}$
-452	$\frac{2^4 11^6 17^3 29^1}{5^3 7^8}$	$\frac{2^8 11^4 17^2 31^4 113^1}{5^6 7^8 29^2}$
-115	$\frac{2^9 3^3}{13^2 23^1}$	$\frac{2^1 5^2 11^2}{13^2 23^1}$
-472	$\frac{3^9 5^4 29^3}{2^1 23^2 31^2 47^2}$	$\frac{19^2 59^1 73^2 113^2}{2^1 5^2 23^2 31^2 47^2}$
-488	$\frac{11^9 17^3}{2^4 13^4 47^2}$	$\frac{11^4 13^2 17^4}{2^4 5^6 47^2}$
-123	$\frac{2^{15} 3^6 5^1}{7^4 23^2}$	$\frac{2^2 13^4 59^2}{5^2 7^4 23^2}$

Table 8.4: Norms of CM Points on $\mathcal{X}_{10}^{*}$ for $0 < -d \leq 250$

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert(2-t_{10})(\mathcal{P}_\Delta)\rvert$
-520	$\frac{3^3 29^3}{2^3 7^2 13^1 47^2}$	$\frac{5^4 11^2 17^2}{2^3 7^2 13^1 47^2}$
-532	$\frac{2^2 3^6 5^6 11^3 23^1}{7^2 29^2 37^2 53^2}$	$\frac{2^{10} 11^2 19^2 113^2}{7^2 23^2 29^2 37^2}$
-548	$\frac{2^4 11^6 29^3 41^3}{5^3 7^8 13^2 53^2}$	$\frac{2^8 11^4 13^4 19^4 137^1}{5^6 7^8 53^2}$
-552	$\frac{3^{15} 5^2 41^3}{2^6 13^6 31^2}$	$\frac{19^4 31^2 59^2}{2^6 5^4 13^4}$
-568	$\frac{3^6 17^3 23^1 41^3}{5^4 7^4 47^2}$	$\frac{17^2 31^2 71^1 97^2 137^2}{5^4 7^4 23^2 47^2}$
-580	$\frac{2^2 3^6 41^3}{13^2 29^1 53^2}$	$\frac{2^4 5^7}{29^1 53^2}$
-155	$\frac{2^{12} 11^3}{7^4 31^1}$	$\frac{2^2 5^6 11^2}{7^4 31^1}$
-628	$\frac{2^3 3^9 5^4 11^3 47^3}{29^2 31^2 53^2 61^2}$	$\frac{2^{13} 11^4 19^2 137^2 157^1}{5^2 29^2 31^2 53^2 61^2}$
-632	$\frac{5^2 11^6 41^3 47^1}{2^2 7^8 13^4}$	$\frac{11^4 17^6 79^1 113^2}{2^2 5^4 7^8 47^2}$
-163	$\frac{2^9 3^3 5^3 11^3}{7^2 13^2 29^2 31^2}$	$\frac{2^1 19^2 59^2 79^2}{7^2 13^2 29^2 31^2}$
-660	$\frac{2^2 3^9 47^3}{7^4 23^2 61^2}$	$\frac{2^8 5^4 11^2 17^2}{7^4 23^2 61^2}$
-680	$\frac{11^3 17^3 41^3}{2^4 7^6 23^2}$	$\frac{5^{10} 11^4}{2^4 7^6 23^2}$
-692	$\frac{2^7 17^6 47^3}{5^4 23^1 31^2 53^2}$	$\frac{2^{27} 17^6 31^2 173^1}{5^{10} 23^4 53^2}$
-708	$\frac{2^2 3^6 5^6 17^3 41^3 53^3}{7^4 13^2 23^4 29^2 37^2 61^2}$	$\frac{2^4 11^4 59^2 97^2 157^2}{7^4 23^4 29^2 37^2 61^2}$
-712	$\frac{3^{12} 47^1 53^3}{2^7 5^3 71^2}$	$\frac{19^4 53^2 79^2 173^2}{2^7 5^6 47^2 71^2}$
-728	$\frac{17^6 29^3 53^3}{2^4 5^2 7^6 13^4 71^2}$	$\frac{17^4 19^4 59^4 137^2}{2^4 5^8 7^6 13^2 71^2}$
-740	$\frac{2^4 11^6 53^3}{23^4 29^2 37^1}$	$\frac{2^8 5^8 11^4}{23^4 29^2}$
-187	$\frac{2^{12} 3^6 5^1 11^3}{23^2 31^2 37^2}$	$\frac{2^2 13^4 71^2}{5^2 23^2 37^2}$

Table 8.4: Norms of CM Points on $\mathcal{X}_{10}^*$ for $0 < -d \leq 250$

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert (2-t_{10})(\mathcal{P}_\Delta)\rvert$
-760	$\frac{3^3 17^3 47^3}{7^2 31^2 71^2}$	$\frac{5^2 11^2 13^2 19^1 37^2}{7^2 31^2 71^2}$
-772	$\frac{2^2 3^6 5^1 29^3 41^3 53^1}{7^4 13^2 37^2 61^2}$	$\frac{2^4 17^2 113^2 173^2 193^1}{5^2 7^4 53^2 61^2}$
-195	$\frac{2^{12} 3^6}{13^2 29^2}$	$\frac{2^2 5^6 19^2}{13^2 29^2}$
-788	$\frac{2^5 17^6 47^3 59^3}{7^{10} 13^4 23^1}$	$\frac{2^{21} 13^2 17^2 19^2 59^4 197^1}{5^6 7^{10} 23^4}$
-808	$\frac{3^9 5^4 41^3 59^3}{2^2 13^2 23^2 47^2 71^2 79^2}$	$\frac{11^6 13^2 157^2 197^2}{2^2 5^2 23^2 47^2 71^2 79^2}$
-203	$\frac{2^{27} 5^2 11^3}{7^4 13^4 37^2}$	$\frac{2^4 11^6 79^2}{5^4 7^4 37^2}$
-820	$\frac{2^2 3^6 59^3}{7^4 31^2 37^2}$	$\frac{2^8 5^7}{7^4 31^2}$
-840	$\frac{3^6 23^3 53^3}{2^4 7^2 13^4 79^2}$	$\frac{5^4 11^4 17^2 19^2}{2^4 7^2 13^4 79^2}$
-852	$\frac{2^4 3^{12} 5^2 11^3 47^3 59^3}{13^6 23^6 61^2}$	$\frac{2^{12} 11^6 19^4 71^2 193^2}{5^4 13^4 23^6 61^2}$
-868	$\frac{2^4 3^{12} 5^2 53^1}{7^4 29^1 37^2}$	$\frac{2^8 31^2 137^2 197^2}{5^4 7^4 29^4}$
-872	$\frac{11^6 17^9 59^3}{2^4 7^{10} 29^4 71^2}$	$\frac{11^6 19^2 31^4 71^2 173^2}{2^4 5^6 7^{10} 29^4}$
-888	$\frac{3^{18} 41^3 47^3}{2^4 5^2 29^4 31^2 79^2}$	$\frac{31^2 37^2 59^4 71^2 97^2}{2^4 5^8 29^4 79^2}$
-227	$\frac{2^{33} 17^3}{13^6 31^2}$	$\frac{2^5 17^4 37^4}{5^6 13^2 31^2}$
-920	$\frac{23^2 59^3}{2^3 29^1 47^2}$	$\frac{5^{15} 37^2}{2^3 23^1 29^4 47^2}$
-932	$\frac{2^6 17^3 23^2 41^3 53^3}{5^2 7^{12} 31^4}$	$\frac{2^{12} 17^6 19^4 53^2 71^4 233^1}{5^8 7^{12} 23^4 31^4}$
-235	$\frac{2^6 3^3 17^3}{7^2 37^2 47^1}$	$\frac{2^1 5^2 11^2 19^2}{7^2 37^2 47^1}$
-948	$\frac{2^6 3^{21} 59^3 71^3}{5^2 31^4 37^2 47^2 61^2}$	$\frac{2^{26} 19^4 79^2 157^2}{5^8 37^2 47^2 61^2}$
-952	$\frac{3^{12} 5^2 71^1}{2^2 7^4 23^1 79^2}$	$\frac{17^2 113^2 193^2 233^2}{2^2 5^4 7^4 23^4 71^2}$

Appendix A

Mathematica Code and Output

This code was implemented in Mathematica (version 5.2) to produce the calculations presented in Chapters 7 and 8. The output that follows corresponds to the input given in the first two lines. The author apologizes for the state of this code as no attempt has been made to make it efficient or properly commented. It is provided solely for completeness.

```
DiscriminantOfQuaternionAlgebra = 10;
DiscriminantOfCMPoint = 52;
DB = DiscriminantOfQuaternionAlgebra;
{tk2c, Ds2, tk3c, Ds3, c2} =
    Switch[DB, 10, {3, 3, -2, 2, 2^(2)}, 6, {-6, 3, 4, 1, 6^(-6)},
      14, {2, 2, -4, 1, 1}];
<< NumberTheory`NumberTheoryFunctions`
<< NumberTheory`AlgebraicNumberFields`
<< SmithForm`
Off[General::spell1]
Squarefree[xxx_] := Module[{divx},
    divx = FactorInteger[xxx];
```

```
    Product[divx[[i, 1]]^Mod[divx[[i, 2]], 2], {i, 1, Length[divx]}]
    ]
d = Abs[If[Mod[DiscriminantOfCMPoint, 4] == 0,
        DiscriminantOfCMPoint/4, DiscriminantOfCMPoint]];
DK = If[Mod[Squarefree[d], 4] == 3, Squarefree[d], 4Squarefree[d]];
hk = ClassNumber[-DK];
PrimeDivisors[xxx_] := If[xxx == 1, {},
    Transpose[FactorInteger[xxx]][[1]]]
TwoAdicReduce[xxx_] :=
    Module[{minpoly, X, rts, rt1, rt2, rat, irrat},
    minpoly = MinimalPolynomial[xxx, X];
    rts = X /. Solve[minpoly == 0, X];
    {rt1, rt2} = If[Length[rts] == 2, rts, Append[rts, rts[[1]]]];
    rat = FullSimplify[(rt1 + rt2)/2];
    irrat = FullSimplify[(rt1 - rt2)/2];
    ratmult =
      If[irrat == 0, 0,
        FullSimplify[Sqrt[irrat^2/
            Squarefree[FullSimplify[irrat^2]]]]];
    FullSimplify[rat + ratmult]
    ]
ord[ppp_, ttt_] := Module[{fac, t = 0, i, tt},
    If[PrimeQ[ppp],
```

```
      If[ttt == 0, t = Infinity,
        tt = If[ppp == 2, TwoAdicReduce[ttt], ttt];
        fac = FactorInteger[tt];
        For[i = 1, i < Length[fac] + 1,
          If[fac[[i, 1]] == ppp, t = fac[[i, 2]]];
          i++], Print["First input ", ppp , " not Prime"];
          t = Null
        ]];
    t
    ]
RamifiedPrimes = PrimeDivisors[DK];
Divides[xxx_, yyy_] := If[Mod[yyy, xxx] == 0, True, False]
stan[xxx_, ppp_] := {xxx/ppp^ord[ppp, xxx], ord[ppp, xxx]}
CharZp[xxx_, ppp_] := If[ord[ppp, xxx] >= 0, 1, 0]
psi[xxx_] := E^(2 Pi I * 2^stan[xxx, 2][[2]])
SubsetOf[xxx_, yyy_] := If[Intersection[xxx, yyy] == xxx,
        True, False]
ModifiedJacobi[xxx_, yyy_] := Module[{twoparty, resty},
    If[Mod[xxx, 4] > 1 && Divides[2, yyy],
      Print[{"UNDEFINED!!", xxx, yyy}],
      resty = yyy/(2^ord[2, yyy]);
      twopartJac =
        If[Divides[2,
```

```
            yyy], (If[Mod[xxx, 4] == 0, 0,
               If[Mod[xxx, 8] == 1, 1, -1]])^ord[2, yyy], 1];
      JacobiSymbol[xxx, resty]*twopartJac]]
reduce[xxx_, ppp_] := Module[{denx, numx},
    denx = Denominator[xxx];
    numx = Numerator[xxx];
    numx*PowerMod[denx, -1, ppp]
    ]
epsi[uuu_] := If[Mod[uuu, 2] == 1, Mod[(uuu - 1)/2, 2], Null];
omeg[uuu_] := If[Mod[uuu, 2] == 1, Mod[(uuu^2 - 1)/8, 2], Null];
Hilbert[xxx_, yyy_, ppp_] := Module[{alpha, beta, u, v, result},
    alpha = ord[ppp, xxx];
    beta = ord[ppp, yyy];
    If[ppp == 2,
      u = reduce[xxx/(ppp^alpha), 8];
      v = reduce[yyy/(ppp^beta), 8],
      u = reduce[xxx/(ppp^alpha), ppp];
      v = reduce[yyy/(ppp^beta), ppp]];
    result =
      If[ppp ==
          2, (-1)^(epsi[u]*epsi[v] + alpha*omeg[v] +
              beta*omeg[u]), (-1)^(alpha*beta*epsi[ppp])*
          ModifiedJacobi[u, ppp]^beta*ModifiedJacobi[v, ppp]^alpha];
```

```
    result
    ]
rhop[ppp_, ttt_] := If[ord[ppp, ttt] >= 0,
    If[ModifiedJacobi[-d, ppp] == 0, 1,
      Sum[(ModifiedJacobi[-d, ppp])^k, {k, 0, ord[ppp, ttt]}]], 0]
rho[ttt_] := If[(ttt \[Element] Integers),
    Module[{divt, i},
      divt = PrimeDivisors[ttt];
      Product[rhop[divt[[i]], ttt], {i, 1, Length[divt]}]
      ],
    0]
WO[ddd_, lis_] := Module[{res = ddd, i},
    For[i = 1, i <= Length[lis],
      res = res/lis[[i]]^ord[lis[[i]], ddd];
      i++];
    res]
InL[mu_, ppp_] := ord[ppp, mu[[1]]] >= 0 && ord[ppp, mu[[2]]] >= 0
chi[xxx_, ppp_] := If[ord[ppp, xxx] == 0, Hilbert[xxx, ppp, ppp], 0]
Present[li_] := Module[{i, top = "", bot = ""},
    For[i = 1, i <= Length[li],
      If[li[[i, 2]] > 0,
        top = top. ToString[li[[i]]],
        bot = bot. ToString[li[[i]]]];
```

```
      i++;];
    Print[top];
    Print["------------------------"];
    Print[bot];
    ]
ExpandLog[xx_] := Module[{fac = FactorInteger[xx]},
      Sum[fac[[i, 2]]*Log[fac[[i, 1]]], {i, 1, Length[fac]}]];
genus[Delt_, Dee_] :=
    2^(Length[PrimeDivisors[GCD[Delt, Dee]]] -
          If[SubsetOf[PrimeDivisors[Delt],
             PrimeDivisors[Dee]], 1, 0]);
SizeOfZU = hk/genus[DK, DB];
q = 2;
For[i = 1, i < 1000,
    Module[{pis, cond = 1},
        pis = PrimeDivisors[DB];
        If[Not[Mod[q, 8] == 5] || Mod[DB, q] == 0, q = NextPrime[q],
          For[j = 1, j <= Length[pis],
            If[pis[[j]] == 2, If[Not[Mod[q, 8] == 5], cond = -1],
              If[JacobiSymbol[q, pis[[j]]] != -1, cond = -1]];
            j++];
          If[cond == -1, q = NextPrime[q], i = 10001];
          ];;]
```

```
      i++];
If[JacobiSymbol[DB, q] == 1,
  m = Position[Table[Mod[i^2, q], {i, 1, q}], Mod[DB, q]][[1, 1]];
  l = Mod[PowerMod[2*DB, -1, q]*m, q];
  mp = 2*DB*l;]
iota[{{aaa_, bbb_}, {ccc_, ddd_}}] := {{ddd, -bbb}, {-ccc, aaa}}
inn[xxx_, yyy_] := FullSimplify[Tr[xxx.iota[yyy]]]
Q[xxx_] := FullSimplify[inn[xxx, xxx]/2]
o = {{1, 0}, {0, 1}};
b = {{Sqrt[q], 0}, {0, -Sqrt[q]}};
c = {{0, DB}, {1, 0}};
bc = b.c;
bcBasis[{{www_, xxx_}, {yyy_, zzz_}}] :=
  FullSimplify[{(www + zzz)/
        2, (www - zzz)/(2 Sqrt[q]), (xxx + DB yyy)/(2DB), (xxx -
            DB yyy)/(2 DB Sqrt[q])}]
e1 = (o + b)/2;
e2 = (mp b + bc)/q;
e3 = e1.e2; L1 =
  FullSimplify[
      bcBasis[A o + (-2 A - D mp) e1 + C e2 + D e3]
          /. {A -> -1, C -> 0, D -> 0}].{o, b, c, bc};
L2 = FullSimplify[
```

```
        bcBasis[A o + (-2 A - D mp) e1 + C e2 + D e3]
            /. {A -> 0, C -> 1, D -> 0}].{o, b, c, bc};
L3 = FullSimplify[
        bcBasis[A o + (-2 A - D mp) e1 + C e2 + D e3]
            /. {A -> 0, C -> 0, D -> 1}].{o, b, c, bc};
L3 = (q - 1)L2/2 + L3;
LBasis[xxx_] :=
  FullSimplify[
    bcBasis[xxx].{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, mp, -q, 2},
          {0, -mp, q, 0}}]
eBasis[xxx_] := LBasis[xxx];
li = 80;
For[i = 0, i < li,
    IT = FullSimplify[
        Reduce[(Det[A L1 + B L2 + C L3] /. {C -> i}) == d,
          {A, B}, Integers, Backsubstitution -> True]];
    If[IT =!= False, Ctemp = i; i = li + 999];
    If[i <= 0, i = -i + 1, If[i < li, i = -i]];
    ];
Z = A L1 + B L2 + Ctemp L3 /.
      ToRules[Reduce[LogicalExpand[IT /. {C[1] -> 0}][[1]], {A, B},
          Integers]];
{z0, z1, z2, z3} = LBasis[Z];
```

```
k =.;i =.;j =.;

k = k /. Solve[inn[Z, i*L1 + j*L2 + k*L3] == 0, k][[1]];

U2 = Coefficient[i*XX + j*YY + k*ZZ, i]

   /. {XX -> L1, YY -> L2, ZZ -> L3};

U1 = Coefficient[i*XX + j*YY + k*ZZ, j]

   /. {XX -> L1, YY -> L2, ZZ -> L3};

U1 = U1*DB q (2 z2 + (-1 + q) z3);

U2 = U2 - inn[U2, U1]/inn[U1, U1]*U1;

U2 = FullSimplify[

      U2*(-4 mp^2 (-1 + q) q^2 z1^2 -

              8 mp (DB + mp^2 (-1 + q)) q z1 z2 +

              4 (DB - mp^2) (DB + mp^2 (-1 + q)) z2^2 +

              4 DB (DB + mp^2 (-1 + q)) q z2 z3 +

              DB (DB + mp^2 (-1 + q)) (-1 + q) q z3^2)/(2 q)];

k =.;i =.;j =.;

alp1 = DB q (2 z2 + (-1 + q) z3)/(2q);

alp2 = -4 q (q z1 + mp z2)/(2q);

alp3 = (-4 DB z2 + 4 mp (q z1 + mp z2) - 2 DB q z3)/(2q);

g123 = GCD[alp1, alp2, alp3];

g12 = GCD[alp1, alp2]/g123;

g13 = GCD[alp1, alp3]/g123;

g23 = GCD[alp2, alp3]/g123;

If[g12 == 0, g12 = 1];
```

```
If[g13 == 0, g13 = 1];

If[g23 == 0, g23 = 1];

alp11 = alp1/(g12 g13 g123);

alp21 = alp2/(g12 g23 g123);

alp31 = alp3/(g13 g23 g123);

{rr1, ss1} = ExtendedGCD[alp11, alp31][[2]];

{rr3, ss3} = ExtendedGCD[alp11 g12, -2 alp11 g12 - alp31 g23][[2]];

Lm1 = alp11 g12 L2 + alp31 g23 L3;

Lm2 = g13 L1 + ss1 alp21 g12 L2 - alp21 g23 rr1 L3;

If[GCD[Denominator[inn[Lm1, Lm2]/inn[Lm2, Lm2]],

        Denominator[inn[Lm1, Lm2]/inn[Lm1, Lm1]]] > 2,

        Lm1 = Lm1 + Lm2];

LmBasis[xxx_] := Module[{A, B, Zero, i, j, k, lis},

    Zero = eBasis[A*Lm1 + B*Lm2 - xxx];

    i = If[eBasis[Lm1][[2]] == 0,

       If[eBasis[Lm1][[3]] == 0, 4, 3], 2];

    lis = Complement[{2, 3, 4}, {i}];

    j = If[eBasis[Lm2][[lis[[1]]]] == 0, lis[[2]], lis[[1]]];

    k = Complement[{2, 3, 4}, {i, j}][[1]];

    A = A /. Solve[Zero[[i]] == 0, A][[1]];

    B = B /. Solve[Zero[[j]] == 0, B][[1]];

    If[FullSimplify[Zero[[k]]] == 0, {A, B}, Print["Nope", {A, B}];]]

CreateOrtho[ppp_] := Module[{num, den, Ax, Bx, IT, i, j},
```

```
OL1 =.;OL2 =.;TOL =.;
den = Denominator[inn[Lm1, Lm2]/inn[Lm1, Lm1]];
num = Numerator[inn[Lm1, Lm2]/inn[Lm1, Lm1]];
If[MemberQ[PrimeDivisors[den], ppp],
  den = Denominator[inn[Lm1, Lm2]/inn[Lm2, Lm2]];
  num = Numerator[inn[Lm1, Lm2]/inn[Lm2, Lm2]];
  If[MemberQ[PrimeDivisors[den], ppp],
    Ax =.;
    Bx =.;
    OL1 = Lm1;
    OL2 = Ax*Lm1 + Bx*Lm2;
    i =.; j =.;
    IT = Q[i*OL1 + j*OL2];
    Ax =
      Ax /. Solve[Coefficient[IT, i j, 1] ==
            Coefficient[IT, i, 2], Ax][[1]];
    Bx =
      Bx /. Solve[Coefficient[IT, j, 2] == Coefficient[IT, i, 2],
            Bx][[1]];
    TOL = FullSimplify[{{1, -Ax/Bx}, {0, 1/Bx}}];
    diag = {Coefficient[IT, i, 2], Coefficient[IT, i, 2]};,
    OL1 = Lm2;
    OL2 = den*Lm1 - num*Lm2;
```

```
        TOL = {{num/den, 1}, {1/den, 0}};
        diag =
          Module[{i, j}, {Coefficient[Q[i*OL1 + j*OL2], i, 2],
              Coefficient[Q[i*OL1 + j*OL2], j, 2]}]];,
      OL1 = Lm1;
      OL2 = den*Lm2 - num*Lm1;
      TOL = {{1, num/den}, {0, 1/den}};
      diag =
        Module[{i, j}, {Coefficient[Q[i*OL1 + j*OL2], i, 2],
            Coefficient[Q[i*OL1 + j*OL2], j, 2]}];
      ];
    ]
OLBasis[xxx_] := Transpose[TOL.Transpose[{LmBasis[xxx]}]][[1]];
epiel[iii_, ppp_] := stan[diag[[iii]], ppp];
CreateOrtho[3];
tempdiag = diag;
If[Not[PrimeDivisors[Abs[Det[TOL]]] == {2}],
  pri = PrimeDivisors[Det[TOL]][[-1]];
  CreateOrtho[pri];]
LatPrimes =
    PrimeDivisors[Abs[tempdiag[[1]]*tempdiag[[2]]*
       diag[[1]]*diag[[2]]]];
diag =.;
```

```
i =.;K1 =.;K2 =.;Y =.;L =.;
Te = K1*U1 + K2*U2;
Ite = Collect[inn[Te, i*Lm1 + j*Lm2], {i, j}];
If[Coefficient[Ite, i K1] != 0,
    K1 = K1 /. Solve[Coefficient[Ite, i] == L, K1][[1]];
    K2 = K2 /. Solve[Coefficient[Ite, j] == Y, K2][[1]],
    K1 = K1 /. Solve[Coefficient[Ite, j] == L, K1][[1]];
    K2 = K2 /. Solve[Coefficient[Ite, i] == Y, K2][[1]]];
{K1, K2};
TeB = eBasis[Te];
TeBY = TeB /. {Y -> 1, L -> 0};
TeBL = TeB /. {L -> 1, Y -> 0};
mvBasis =
    Developer`HermiteNormalForm[{{TeBY[[2]], TeBY[[3]],
            TeBY[[4]]}, {TeBL[[2]], TeBL[[3]], TeBL[[4]]}}][[2]];
Lmv1 = (mvBasis[[1]].{{L1}, {L2}, {L3}})[[1]];
Lmv2 = (mvBasis[[2]].{{L1}, {L2}, {L3}})[[1]];
Zv = Z/(2d);
RelMat = {LBasis[Z][[{2, 3, 4}]], LBasis[Lm1][[{2, 3, 4}]],
      LBasis[Lm2][[{2, 3, 4}]]};
CosetRep[RM_] := Module[{SS, UU, VV},
      {SS, {UU, VV}} = IntegerSmithForm[RM];
      VV = Inverse[VV];
```

```
      If[Abs[SS[[3, 3]]] == Abs[Det[RM]], Cycl = 1;
        VV[[3, 1]]*L1 + VV[[3, 2]]*L2 + VV[[3, 3]]*L3,
        Cycl = 0;
        {{VV[[3, 1]]*L1 + VV[[3, 2]]*L2 + VV[[3, 3]]*L3, SS[[3, 3]]},
        {VV[[2, 1]]*L1 + VV[[2, 2]]*L2 + VV[[2, 3]]*L3, SS[[2, 2]]},
        {VV[[1, 1]]*L1 + VV[[1, 2]]*L2 + VV[[1, 3]]*L3, SS[[1, 1]]}}]
      ];
CosRep = CosetRep[RelMat];
i =.;
inn[i*Zv, Z] == i;
LamLim = Abs[Det[RelMat]];
lam[nnn_] :=
    If[Length[CosRep] == 2, nnn*CosRep,
      Floor[nnn/(CosRep[[1, 2]]*CosRep[[2, 2]])]*L1 +
        Floor[Mod[nnn,
         (CosRep[[1, 2]]*CosRep[[2, 2]])]/(CosRep[[1, 2]])]*L2 +
         Mod[nnn, CosRep[[1, 2]]]*L3];
plus[xxx_] := inn[xxx, Z]/inn[Z, Z]*Z
minus[xxx_] := xxx - plus[xxx]
lamplus[nnn_] := plus[lam[nnn]]
lamminus[nnn_] := minus[lam[nnn]]
ModL[xxx_] := Module[{bas},
    bas = Mod[eBasis[xxx], 1];
```

```
    bas[[1]]*o + bas[[2]]*c + bas[[3]]*bc + bas[[4]]*e]

EtaPlus[eta_] := plus[ModL[eta]]

EtaMinus[eta_] := minus[ModL[eta]]

kappalist[eta_, mmm_, lim_] :=

  Module[{QvalP, QvalM, top, X, last, etam, etap, lamp, lamm, n, i,

      templist = {}},

    etap = EtaPlus[eta];

    etam = EtaMinus[eta];

    For[n = 0, n < LamLim,

      lamp = lamplus[n];

      lamm = lamminus[n];

      top = Round[X /. Solve[D[Q[lamp + etap + X*Z], X] == 0,

          X][[1]]];

      For[i = 0, i <= lim,

        QvalP = mmm - Q[lamp + etap + (top + (1 + i))*Z];

        If[QvalP >= 0 ,

          templist = Append[templist,

             {QvalP, (lamm + etam), n, (top + (1 + i))}];

          ];

        QvalM = mmm - Q[lamp + etap + (top - i)*Z];

        If[QvalM >= 0 ,

          templist = Append[templist,

             {QvalM, (lamm + etam), n, (top - i)}];
```

```
        ];
        If[QvalM < 0 && QvalP < 0, i = lim];
        i++];
      n++];
    templist]
Hmu[mu_, ppp_] := Module[{res = {}},
    If[CharZp[mu[[1]], ppp] == 1, res = Append[res, 1]];
    If[CharZp[mu[[2]], ppp] == 1, res = Append[res, 2]];
    res]
Kmu[mu_, ppp_] :=
  If[CharZp[mu[[1]], ppp] == 1 && CharZp[mu[[2]], ppp] == 1,
    Infinity,
    Module[{nothmu, i},
      nothmu = Complement[{1, 2}, Hmu[mu, ppp]];
      Min[
        Table[epiel[nothmu[[i]], ppp][[2]] +
            ord[ppp, mu[[nothmu[[i]]]]],
            {i, 1, Length[nothmu]}]]]]
Lmu[mu_, kkk_, ppp_] := Module[{it, h, i, res = {}},
    h = Hmu[mu, ppp];
    For[i = 1, i <= Length[h],
      it = epiel[h[[i]], ppp][[2]] - kkk;
      If[it < 0 && Mod[it, 2] == 1, res = Append[res, h[[i]]]];
```

```
      i++];
    res]
lmu[mu_, kkk_, ppp_] := Length[Lmu[mu, kkk, ppp]]
dmu[mu_, kkk_, ppp_] :=
  kkk + (1/2)*
      Sum[Min[{epiel[Hmu[mu, ppp][[i]], ppp][[2]] - kkk, 0}], {i, 1,
          Length[Hmu[mu, ppp]]}]
epimu[mu_, kkk_, ppp_] :=
  Hilbert[-1, ppp, ppp]^IntegerPart[lmu[mu, kkk, ppp]/2]*
    Module[{li, i},
      li = Lmu[mu, kkk, ppp];
      Product[
        Hilbert[epiel[li[[i]], ppp][[1]], ppp, ppp],
        {i, 1, Length[li]}]]
tmu[mu_, mmm_, ppp_] := Module[{nothmu, res = 0, ep, i},
    nothmu = Complement[{1, 2}, Hmu[mu, ppp]];
    For[i = 1, i <= Length[nothmu],
      res = res + diag[[nothmu[[i]]]]*mu[[nothmu[[i]]]]^2;
      i++];
    mmm - res]
aye[mu_, mmm_, ppp_] := ord[ppp, tmu[mu, mmm, ppp]]
f1[mu_, mmm_, ppp_] :=
  If[Mod[lmu[mu, ord[ppp, mmm] + 1, ppp], 2] == 0, -1/ppp,
```

```
    Hilbert[stan[mmm, ppp][[1]], ppp, ppp]/Sqrt[ppp]]

det2S[ppp_] := ppp^(-ord[ppp, 4*diag[[1]]*diag[[2]]]/2)

Lp[ppp_, XXX_] := (1 -

        If[Divides[ppp, DK], 0, Hilbert[ppp, -DK, ppp]]*XXX/ppp)^(-1)

Wmp[mmm_, ppp_, mu_, XXX_] := Module[{},

    CreateOrtho[ppp];

    nmu = OLBasis[mu];

    If[ppp == 2, Wm2[mmm, nmu, XXX],

      det2S[ppp]*Lp[ppp, XXX]*

        If[aye[nmu, mmm, ppp] >= Kmu[nmu, ppp],

          Expand[(1 + (1 - 1/ppp)*

                  Sum[If[Mod[lmu[nmu, k, ppp], 2] == 0,

                      epimu[nmu, k, ppp]*ppp^dmu[nmu, k, ppp]*

                         XXX^k, 0],

                      {k, 1, Kmu[nmu, ppp]}])],

          If[aye[nmu, mmm, ppp] >= 0,

            Expand[(1 + (1 - 1/ppp)*

                    Sum[If[Mod[lmu[nmu, k, ppp], 2] == 0,

                        epimu[nmu, k, ppp]*ppp^dmu[nmu, k, ppp]*

                           XXX^k, 0],

                        {k, 1, aye[nmu, mmm, ppp]}] +

                  epimu[nmu, aye[nmu, mmm, ppp] + 1, ppp]*

                    f1[nmu, tmu[nmu, mmm, ppp], ppp]*
```

```
                    ppp^dmu[nmu, aye[nmu, mmm, ppp] + 1, ppp]*
                    XXX^(aye[nmu, mmm, ppp] + 1))], 1]]]
    ]
Wmp0[mmm_, ppp_, mu_] := Wmp[mmm, ppp, mu, 1]
DWmp0[mmm_, ppp_, mu_] := Module[{sss},
    If[Wmp0[mmm, ppp, mu] == 0,
      D[Wmp[mmm, ppp, mu, ppp^(-sss)], sss] /. {sss -> 0}]]
ord2[mu_] := Min[{ord[2, mu[[2]]], ord[2, mu[[1]]]}]
Nmu2[mu_] := If[InL[mu, 2], {1}, {}]
dmu2[mu_, kkk_] :=
  kkk + Sum[Min[ord[2, diag[[1]]] - kkk, 0], {i, 1, Length[Nmu[mu]]}]
pmu2[mu_, kkk_] := (-1)^Sum[ord[2, diag[[1]]] - kkk,
    {i, 1, Length[Nmu[mu]]}]
tmu2[mu_, mmm_] := If[InL[mu, 2], mmm, 0]
numu2[mu_, mmm_, kkk_] := tmu2[mu, mmm]*2^(3 - kkk)
aye2[mu_, mmm_] := ord[2, tmu2[mu, mmm]]
Kmu2[mu_, mmm_] :=
  If[InL[mu, 2], aye2[mu, mmm] + 4, ord[2, diag[[1]]] + ord2[mu]]
dmu2d[mu_, kkk_] :=
  kkk + (1/2)*
      Sum[Min[{epiel[Hmu[mu, 2][[i]], 2][[2]] - kkk + 1, 0}], {i, 1,
          Length[Hmu[mu, 2]]}]
epimu2[mu_, kkk_] := Module[{li, i},
```

```
    li = Lmu[mu, kkk - 1, 2];
    Product[epiel[li[[i]], 2][[1]], {i, 1, Length[li]}]]
deltamu2[mu_, kkk_] := Module[{tem},
    tem =
      Product[epiel[Hmu[mu, 2][[i]], 2][[2]] - kkk + 1, {i, 1,
          Length[Hmu[mu, 2]]}];
    If[tem == 0, 0, 1]]
tmu2d[mu_, mmm_] := Module[{nothmu, res = 0, ep, i},
    nothmu = Complement[{1, 2}, Hmu[mu, 2]];
    For[i = 1, i <= Length[nothmu],
      res = res + diag[[nothmu[[i]]]]*mu[[nothmu[[i]]]]^2;
      i++];
    mmm - res]
numu2d[mu_, mmm_, kkk_] :=
  tmu2d[mu, mmm]*2^(3 - kkk) -
    Sum[If[epiel[Hmu[mu, 2][[i]], 2][[2]] < kkk - 1,
        epiel[Hmu[mu, 2][[i]], 2][[1]], 0],
        {i, 1, Length[Hmu[mu, 2]]}]
aye2d[mu_, mmm_] := ord[2, tmu2d[mu, mmm]]
Kmu2d[mu_, mmm_] := If[InL[mu, 2], aye2d[mu, mmm] + 30,
    Module[{i, lis = {}},
      For[i = 1, i <= 2,
        If[ord[2, mu[[i]]] < -1,
```

```
        lis = Append[lis, epiel[i, 2][[2]] + ord[2, mu[[i]]] + 1]];
      If[ord[2, mu[[i]]] == -1,
        lis = Append[lis, epiel[i, 2][[2]] + 1]];
      i++];
    Min[lis]]]
Wm2[mmm_, mu_, XXX_] := Module[{iji, jij},
    If[Coefficient[Q[iji OL1 + jij OL2], iji jij, 1] == 0,
(*Diagonal Formula*)
          Lp[2, XXX]*
      det2S[2]*(1 +
          Sum[If[Mod[lmu[mu, k - 1, 2], 2] == 0,
              deltamu2[mu, k]*2^(dmu2d[mu, k] - 1)*
                chi[epimu2[mu, k], 2]*
                psi[numu2d[mu, mmm, k]/8]*
                CharZp[numu2d[mu, mmm, k]/4, 2]*
                XXX^k, deltamu2[mu, k]*2^(dmu2d[mu, k] - 3/2)*
                chi[epimu2[mu, k]*numu2d[mu, mmm, k], 2]*XXX^k],
              {k, 1, Kmu2d[mu, mmm]}]),
    (*Mixed Formula*)
    Lp[2, XXX]*2^(-ord[2, diag[[1]]^2]/2)*(1 +
          Sum[pmu2[mu, k]*2^(dmu2[mu, k] - 1)*
              psi[numu2[mu, mmm, k]/8]*
              CharZp[numu2[mu, mmm, k]/4, 2]*XXX^k,
```

```
                {k, 1, Kmu2[mu, mmm]}])
      ]]
Wm20[mmm_, mu_] := Module[{XX}, Wm2[mmm, mu, XX] /. {XX -> 1}]
DWm20[mmm_, mu_] := Module[{sss},
    If[Wm20[mmm, mu] == 0, D[Wm2[mmm, mu, 2^(-sss)], sss]
         /. {sss -> 0}]]
ListKappa[kap_] :=
  For[iii = 1, iii <= Length[kap],
    Print[{kap[[iii, 1]], eBasis[kap[[iii, 2]]],
        OLBasis[kap[[iii, 2]]], {Wm20[kap[[iii, 1]],
          OLBasis[kap[[iii, 2]]]],
          DWm20[kap[[iii, 1]], OLBasis[kap[[iii, 2]]]]},
        Table[{Wmp0[kap[[iii, 1]], SpecialPrimes[[i]],
              OLBasis[kap[[iii, 2]]]],
            DWmp0[kap[[iii, 1]], SpecialPrimes[[i]],
              OLBasis[kap[[iii, 2]]]]},
              {i, 1, Length[SpecialPrimes]}]}];
    iii++]
NeededPrimes[mmm_] :=
  Union[Union[PrimeDivisors[Abs[mmm]], LatPrimes], {2}]
W[mmm_, mu_, sss_] :=
  Product[Wmp[mmm, NeededPrimes[mmm][[i]], mu,
      NeededPrimes[mmm][[i]]^(-sss)],
```

```
      {i, 1, Length[NeededPrimes[mmm]]}]
SumKappa[kap_] :=
  Expand[Sum[
      Module[{ss},
        If[kap[[i, 1]] == 0,
          INFINITY, -2*DK^((ss + 1)/2)/hk*
              D[W[kap[[i, 1]], kap[[i, 2]], ss], ss]
                /. {ss -> 0}]], {i, 1, Length[kap]}]]
Print["d = ", d];
Print["Class Number = ", hk];
Print["Z = ", eBasis[Z]];
Print[""];
Print["Basis of Negative Plane:"];
Print["U1 = ", eBasis[U1]];
Print["U2 = ", eBasis[U2]];
X =.; Y =.;
Print["Quadratic Form: ", Q[X U1 + Y U2]]; Print[""];
Print["Basis of Lattice over integers:"]
Print["Lm1 = ", eBasis[Lm1]];
Print["Lm2 = ", eBasis[Lm2]];
X =.; Y =.;
Print["Quadratic Form over this basis: ", Q[X Lm1 + Y Lm2]];
Print[""];
```

```
CreateOrtho[3];

If[Det[TOL] != 1,

  CreateOrtho[2];

  Print["Basis over the 2-adics:"];

  Print["Lm1_2 = ", LmBasis[OL1][[1]], "*Lm1 + ",

    LmBasis[OL1][[2]], "*Lm2"];

  Print["Lm2_2 = ", LmBasis[OL2][[1]], "*Lm1 + ",

    LmBasis[OL2][[2]], "*Lm2"];

  Print["Quadratic Form: ", Q[X OL1 + Y OL2]]; Print[""];

  CreateOrtho[3];

  fc = PrimeDivisors[Det[TOL]];

  If[fc == {2},

    Print["Basis over p-adics for all other p:"];

    Print["Lm1_p = ", LmBasis[OL1][[1]], "*Lm1 + ",

      LmBasis[OL1][[2]], "*Lm2"];

    Print["Lm2_p = ", LmBasis[OL2][[1]], "*Lm1 + ",

      LmBasis[OL2][[2]], "*Lm2"];

    Print["Quadratic Form: ", Q[X OL1 + Y OL2]]; Print[""];,

    Print["Basis over p-adics for p NOT in ", fc, ":"];

    Print["Lm1_p = ", LmBasis[OL1][[1]], "*Lm1 + ",

      LmBasis[OL1][[2]], "*Lm2"];

    Print["Lm2_p = ", LmBasis[OL2][[1]], "*Lm1 + ",

      LmBasis[OL2][[2]], "*Lm2"];
```

```
    Print["Quadratic Form: ", Q[X OL1 + Y OL2]]; Print[""];
    CreateOrtho[fc[[-1]]];
    Print["Basis over p-adics for p in ", Complement[fc, {2}], ":"];
    Print["Lm1_p = ", LmBasis[OL1][[1]], "*Lm1 + ",
      LmBasis[OL1][[2]], "*Lm2"];
    Print["Lm2_p = ", LmBasis[OL2][[1]], "*Lm1 + ",
      LmBasis[OL2][[2]], "*Lm2"];
    Print["Quadratic Form: ", Q[X OL1 + Y OL2]]; Print[""];]]
If[Length[CosRep] == 2,
  Print["L/(L- + L+) generated by lam = ", eBasis[lam[1]]];
Print["and is cyclic of order ", LamLim];
Print["lam- = ", LmBasis[lamminus[1]][[1]], "*Lm1 + ",
    LmBasis[lamminus[1]][[2]], "*Lm2"];
Print["lam+ = ", inn[lamplus[1], Z]/inn[Z, Z], "*Z"];]
k2 = kappalist[0*o, Ds2, 100];
k3 = kappalist[0*o, Ds3, 100];
kp = k2;
Print["TABLE 1:"];
Print[Prepend[
        Table[{kp[[i, 3]], (inn[lamplus[kp[[i, 3]]], Z]/inn[Z, Z] +
                kp[[i, 4]])"Z",
            kp[[i, 1]], -2*DK^((ss + 1)/2)/hk*
                D[W[kp[[i, 1]], kp[[i, 2]], ss], ss] /. {ss -> 0}},
```

```
            {i, 1, Length[kp]}],
       {"i", "x", ToString[Ds2] <> "-Q[x]", "k_u(m)"}] //TableForm];
kp = k3;
Print["TABLE 2:"];
Print[Prepend[
        Table[{kp[[i, 3]], (inn[lamplus[kp[[i, 3]]], Z]/inn[Z, Z] +
                kp[[i, 4]])"Z",
            kp[[i, 1]], -2*DK^((ss + 1)/2)/hk*
                D[W[kp[[i, 1]], kp[[i, 2]], ss], ss] /. {ss -> 0}},
            {i, 1, Length[kp]}],
      {"i", "x", ToString[Ds3] <> "-Q[x]", "k_u(m)"}] //TableForm];
X =.;
Leg = {"p", "u1", "u2", "W^*_{m,p}(s,u)", "W^*_{m,p}(0,u)",
      "d/ds W^*_{m,p}(0,u)"};
kp = k2;
Print["TABLE 3:"];
For[i1 = 1, i1 <= Length[kp],
    pr = NeededPrimes[kp[[i1, 1]]];
    Print["For u = ", LmBasis[kp[[i1, 2]]][[1]], "*Lm1+",
      LmBasis[kp[[i1, 2]]][[2]],
      "*Lm2 = u1*Lm1_p+u2*Lm2_p  and m = ", kp[[i1, 1]]];
    Tb = Table[{pr[[j1]], CreateOrtho[pr[[j1]]];
          OLBasis[kp[[i1, 2]]][[1]],
```

```
        OLBasis[kp[[i1, 2]]][[2]],
        Simplify[Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]], X]],
        Simplify[Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]], X]
           /. {X -> 1}],
        If[Simplify[
              Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]], X]
              /. {X -> 1}] == 0,
          Simplify[
            D[Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]],
                pr[[j1]]^(-X)], X] /. {X -> 0}], ""]},
                {j1, 1, Length[pr]}];
    Print[Prepend[Tb, Leg] // TableForm];
    Print["Thus k_u(", kp[[i1, 1]],
      ") = ", -2*DK^((ss + 1)/2)/hk*
        D[W[kp[[i1, 1]], kp[[i1, 2]], ss], ss] /. {ss -> 0}];
    Print[""];
    i1++];
tk2 = SumKappa[k2];
Print["Summing these gives: kappa_0(", Ds2, ") = ", tk2]
Print[""]; Print[""];
kp = k3;
Print["TABLE 4:"];
For[i1 = 1, i1 <= Length[kp],
```

```
pr = NeededPrimes[kp[[i1, 1]]];

Print["For u = ", LmBasis[kp[[i1, 2]]][[1]], "*Lm1+",

  LmBasis[kp[[i1, 2]]][[2]],

  "*Lm2 = u1*Lm1_p+u2*Lm2_p  and m = ",

  kp[[i1, 1]]];

Tb = Table[{pr[[j1]], CreateOrtho[pr[[j1]]];

      OLBasis[kp[[i1, 2]]][[1]],

      OLBasis[kp[[i1, 2]]][[2]],

      Simplify[Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]], X]],

      Simplify[Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]], X]

         /. {X -> 1}],

      If[Simplify[

            Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]], X]

            /. {X -> 1}] == 0,

        Simplify[

          D[Wmp[kp[[i1, 1]], pr[[j1]], kp[[i1, 2]],

              pr[[j1]]^(-X)], X] /. {X -> 0}], ""]},

              {j1, 1, Length[pr]}];

Print[Prepend[Tb, Leg] // TableForm];

Print["Thus k_u(", kp[[i1, 1]],

  ") = ",  (-2*DK^((ss + 1)/2)/hk*

        D[W[kp[[i1, 1]], kp[[i1, 2]], ss], ss]) /. {ss -> 0}];

Print[""];
```

```
    i1++];

tk3 = SumKappa[k3];

Print["Summing these gives: kappa_0(", Ds3, ") = ", tk3]

Print[""]; Print[""];

Print["To Recap:"];

Print["kappa_0(", Ds2, ") = ", tk2];

Print["kappa_0(", Ds3, ") = ", tk3];

Print["Ideal Class Number: ", hk];

Print["Field Discriminant: ", DK];

Print["Normalizing Constant: C2 = ", c2];

c1 = -SizeOfZU/2^(Length[PrimeDivisors[DB]]);

Print["Final Result:"];

FinRes = Expand[c1*(tk3c*tk3 + tk2c*tk2) - SizeOfZU*ExpandLog[c2]];

Print[c1, "*(", tk2c, "*kappa_0(", Ds2, ")+",

    tk3c, "*kappa_0(", Ds3, "))-", SizeOfZU, "*Log[C2]= ", FinRes];
```

d = 13

Class Number = 2

Z = $\{0, 93, -5, 0\}$

Basis of Negative Plane:

U1 = $\{0, 0, -1300, 8840\}$

U2 = $\{0, 20220200, -1094340, 162240\}$

Quadratic Form: $-5257252000\,(X^2 + 13Y^2)$

Basis of Lattice over integers:

Lm1 = $\{0, 5, 4, -29\}$

Lm2 = $\{0, 5, 9, -63\}$

Quadratic Form over this basis: $-5\,(11347X^2 + 49264XY + 53471Y^2)$

Basis over the 2-adics:

Lm1_2 = 1*Lm1 + 0*Lm2

Lm2_2 = -24632*Lm1 + 11347*Lm2

Quadratic Form: $-56735\,(X^2 + 13Y^2)$

Basis over p-adics for p NOT in $\{7, 1621\}$:

Lm1_p = 1*Lm1 + 0*Lm2

Lm2_p = -24632*Lm1 + 11347*Lm2

Quadratic Form: $-56735\,(X^2 + 13Y^2)$

Basis over p-adics for p in $\{7, 1621\}$:

Lm1_p = 0*Lm1 + 1*Lm2

Lm2_p = 53471*Lm1 + -24632*Lm2

Quadratic Form: $-267355\,(X^2 + 13Y^2)$

L/(L- + L+) generated by lam = $\{0, 0, 0, 1\}$

and is cyclic of order 13

lam- = $\frac{862}{13}$*Lm1 + $-\frac{397}{13}$*Lm2

lam+ = $-\frac{25}{13}$*Z

TABLE 1:

i	x	3-Q[x]	k_u(m)
0	0	3	0
1	$\frac{Z}{13}$	$\frac{38}{13}$	$-\frac{2\text{Log}[5]}{3}$
2	$\frac{2Z}{13}$	$\frac{35}{13}$	$-4\text{Log}[2]$
3	$\frac{3Z}{13}$	$\frac{30}{13}$	$-2\text{Log}[3]$
4	$\frac{4Z}{13}$	$\frac{23}{13}$	0
5	$\frac{5Z}{13}$	$\frac{14}{13}$	$-\frac{2\text{Log}[5]}{3}$
6	$\frac{6Z}{13}$	$\frac{3}{13}$	0
7	$-\frac{6Z}{13}$	$\frac{3}{13}$	0
8	$-\frac{5Z}{13}$	$\frac{14}{13}$	$-\frac{2\text{Log}[5]}{3}$
9	$-\frac{4Z}{13}$	$\frac{23}{13}$	0
10	$-\frac{3Z}{13}$	$\frac{30}{13}$	$-2\text{Log}[3]$
11	$-\frac{2Z}{13}$	$\frac{35}{13}$	$-4\text{Log}[2]$
12	$-\frac{Z}{13}$	$\frac{38}{13}$	$-\frac{2\text{Log}[5]}{3}$

TABLE 2:

i	x	2-Q[x]	k_u(m)
0	0	2	0
1	$\frac{Z}{13}$	$\frac{25}{13}$	$-\frac{7\text{Log}[5]}{3}$
2	$\frac{2Z}{13}$	$\frac{22}{13}$	$-\frac{2\text{Log}[5]}{3}$
3	$\frac{3Z}{13}$	$\frac{17}{13}$	$-\frac{2\text{Log}[5]}{3}$
4	$\frac{4Z}{13}$	$\frac{10}{13}$	$-3\text{Log}[2]$
5	$\frac{5Z}{13}$	$\frac{1}{13}$	$-\frac{\text{Log}[5]}{3}$
8	$-\frac{5Z}{13}$	$\frac{1}{13}$	$-\frac{\text{Log}[5]}{3}$
9	$-\frac{4Z}{13}$	$\frac{10}{13}$	$-3\text{Log}[2]$
10	$-\frac{3Z}{13}$	$\frac{17}{13}$	$-\frac{2\text{Log}[5]}{3}$
11	$-\frac{2Z}{13}$	$\frac{22}{13}$	$-\frac{2\text{Log}[5]}{3}$
12	$-\frac{Z}{13}$	$\frac{25}{13}$	$-\frac{7\text{Log}[5]}{3}$

TABLE 3:

For u = 0*Lm1+0*Lm2 = u1*Lm1_p+u2*Lm2_p and m = 3

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	0	0	$\frac{1}{2}(1-X^2)$	0	Log[2]
3	0	0	$1-X$	0	Log[3]
5	0	0	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	0	0	1	1	
11	0	0	1	1	
13	0	0	$\frac{1+X}{\sqrt{13}}$	$\frac{2}{\sqrt{13}}$	
1621	0	0	1	1	
4861	0	0	1	1	

Thus k_u(3) = 0

For u = $\frac{862}{13}$*Lm1+ $-\frac{397}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{38}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{170}{11347}$	$-\frac{397}{147511}$	$\frac{1}{2}(1+X^3)$	1	
5	$\frac{170}{11347}$	$-\frac{397}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{369}{53471}$	$\frac{862}{695123}$	1	1	
11	$\frac{170}{11347}$	$-\frac{397}{147511}$	1	1	
13	$\frac{170}{11347}$	$-\frac{397}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
19	$\frac{170}{11347}$	$-\frac{397}{147511}$	$1+X$	2	
1621	$\frac{369}{53471}$	$\frac{862}{695123}$	1	1	
4861	$\frac{170}{11347}$	$-\frac{397}{147511}$	1	1	

Thus k_u($\frac{38}{13}$) = $-\frac{2\text{Log}[5]}{3}$

For u = $\frac{1724}{13}$*Lm1+ $-\frac{794}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{35}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{340}{11347}$	$-\frac{794}{147511}$	$\frac{1}{2}\left(1-X^2\right)$	0	Log[2]
5	$\frac{340}{11347}$	$-\frac{794}{147511}$	$\frac{1+4X+X^2}{5+X}$	1	
7	$\frac{738}{53471}$	$\frac{1724}{695123}$	$1+X$	2	
11	$\frac{340}{11347}$	$-\frac{794}{147511}$	1	1	
13	$\frac{340}{11347}$	$-\frac{794}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{738}{53471}$	$\frac{1724}{695123}$	1	1	
4861	$\frac{340}{11347}$	$-\frac{794}{147511}$	1	1	

Thus k_u($\frac{35}{13}$) = $-$ 4Log[2]

For u = $\frac{2586}{13}$*Lm1+ $-\frac{1191}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{30}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$\frac{1}{2}\left(1+X^3\right)$	1	
3	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$1-X$	0	Log[3]
5	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$\frac{1+4X+X^2}{5+X}$	1	
7	$\frac{1107}{53471}$	$\frac{2586}{695123}$	1	1	
11	$\frac{510}{11347}$	$-\frac{1191}{147511}$	1	1	
13	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{1107}{53471}$	$\frac{2586}{695123}$	1	1	
4861	$\frac{510}{11347}$	$-\frac{1191}{147511}$	1	1	

Thus k_u($\frac{30}{13}$) = $-$ 2Log[3]

For u = $\frac{3448}{13}$*Lm1+ $-\frac{1588}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{23}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$\frac{1}{2}(1-X^2)$	0	Log[2]
5	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log[5]}}{6}$
7	$\frac{1476}{53471}$	$\frac{3448}{695123}$	1	1	
11	$\frac{680}{11347}$	$-\frac{1588}{147511}$	1	1	
13	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
23	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$1-X$	0	Log[23]
1621	$\frac{1476}{53471}$	$\frac{3448}{695123}$	1	1	
4861	$\frac{680}{11347}$	$-\frac{1588}{147511}$	1	1	

Thus k_u($\frac{23}{13}$) = 0

For u = $\frac{4310}{13}$*Lm1+ $-\frac{1985}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{14}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{850}{11347}$	$-\frac{1985}{147511}$	$\frac{1}{2}(1+X^3)$	1	
5	$\frac{850}{11347}$	$-\frac{1985}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log[5]}}{6}$
7	$\frac{1845}{53471}$	$\frac{4310}{695123}$	$1+X$	2	
11	$\frac{850}{11347}$	$-\frac{1985}{147511}$	1	1	
13	$\frac{850}{11347}$	$-\frac{1985}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{1845}{53471}$	$\frac{4310}{695123}$	1	1	
4861	$\frac{850}{11347}$	$-\frac{1985}{147511}$	1	1	

Thus k_u($\frac{14}{13}$) = $-\frac{\text{2Log[5]}}{3}$

For u = $\frac{5172}{13}$*Lm1+ $-\frac{2382}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{3}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1020}{11347}$	$-\frac{2382}{147511}$	$\frac{1}{2}(1-X^2)$	0	Log[2]
3	$\frac{1020}{11347}$	$-\frac{2382}{147511}$	$1-X$	0	Log[3]
5	$\frac{1020}{11347}$	$-\frac{2382}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{2214}{53471}$	$\frac{5172}{695123}$	1	1	
11	$\frac{1020}{11347}$	$-\frac{2382}{147511}$	1	1	
13	$\frac{1020}{11347}$	$-\frac{2382}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{2214}{53471}$	$\frac{5172}{695123}$	1	1	
4861	$\frac{1020}{11347}$	$-\frac{2382}{147511}$	1	1	

Thus k_u($\frac{3}{13}$) = 0

For u = $\frac{6034}{13}$*Lm1+ $-\frac{2779}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{3}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{170}{1621}$	$-\frac{397}{21073}$	$\frac{1}{2}(1-X^2)$	0	Log[2]
3	$\frac{170}{1621}$	$-\frac{397}{21073}$	$1-X$	0	Log[3]
5	$\frac{170}{1621}$	$-\frac{397}{21073}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{2583}{53471}$	$\frac{6034}{695123}$	1	1	
11	$\frac{170}{1621}$	$-\frac{397}{21073}$	1	1	
13	$\frac{170}{1621}$	$-\frac{397}{21073}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{2583}{53471}$	$\frac{6034}{695123}$	1	1	
4861	$\frac{170}{1621}$	$-\frac{397}{21073}$	1	1	

Thus k_u($\frac{3}{13}$) = 0

For u = $\frac{6896}{13}$*Lm1+ $-\frac{3176}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{14}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	$\frac{1}{2}(1+X^3)$	1	
5	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{2952}{53471}$	$\frac{6896}{695123}$	$1+X$	2	
11	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	1	1	
13	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{2952}{53471}$	$\frac{6896}{695123}$	1	1	
4861	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	1	1	

Thus k_u($\frac{14}{13}$) = $-\frac{2\text{Log}[5]}{3}$

For u = $\frac{7758}{13}$*Lm1+ $-\frac{3573}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{23}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$\frac{1}{2}(1-X^2)$	0	Log[2]
5	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{3321}{53471}$	$\frac{7758}{695123}$	1	1	
11	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	1	1	
13	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
23	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$1-X$	0	Log[23]
1621	$\frac{3321}{53471}$	$\frac{7758}{695123}$	1	1	
4861	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	1	1	

Thus k_u($\frac{23}{13}$) = 0

For u = $\frac{8620}{13}$*Lm1+ $-\frac{3970}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{30}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$\frac{1}{2}(1+X^3)$	1	
3	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$1-X$	0	Log[3]
5	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$\frac{1+4X+X^2}{5+X}$	1	
7	$\frac{3690}{53471}$	$\frac{8620}{695123}$	1	1	
11	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	1	1	
13	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{3690}{53471}$	$\frac{8620}{695123}$	1	1	
4861	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	1	1	

Thus k_u($\frac{30}{13}$) = -2Log[3]

For u = $\frac{9482}{13}$*Lm1+ $-\frac{4367}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{35}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$\frac{1}{2}(1-X^2)$	0	Log[2]
5	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$\frac{1+4X+X^2}{5+X}$	1	
7	$\frac{369}{4861}$	$\frac{862}{63193}$	$1+X$	2	
11	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	1	1	
13	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{369}{4861}$	$\frac{862}{63193}$	1	1	
4861	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	1	1	

Thus k_u($\frac{35}{13}$) = -4Log[2]

For u = $\frac{10344}{13}$*Lm1+ $-\frac{4764}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{38}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$\frac{1}{2}\left(1+X^3\right)$	1	
5	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{4428}{53471}$	$\frac{10344}{695123}$	1	1	
11	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	1	1	
13	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
19	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$1+X$	2	
1621	$\frac{4428}{53471}$	$\frac{10344}{695123}$	1	1	
4861	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	1	1	

Thus k_u($\frac{38}{13}$) = $-\frac{2\text{Log}[5]}{3}$

Summing these gives: kappa_0(3) = $-8\text{Log}[2]-4\text{Log}[3]-\frac{8\text{Log}[5]}{3}$

TABLE 4:

For u = 0*Lm1+0*Lm2 = u1*Lm1_p+u2*Lm2_p and m = 2

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	0	0	$\frac{1}{2}(1-X^3)$	0	$\frac{3\text{Log}[2]}{2}$
5	0	0	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	0	0	1	1	
11	0	0	1	1	
13	0	0	$-\frac{-1+X}{\sqrt{13}}$	0	$\frac{\text{Log}[13]}{\sqrt{13}}$
1621	0	0	1	1	
4861	0	0	1	1	

Thus k_u(2) = 0

For u = $\frac{862}{13}$*Lm1+ $-\frac{397}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{25}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{170}{11347}$	$-\frac{397}{147511}$	$\frac{1}{2}(1+X^2)$	1	
5	$\frac{170}{11347}$	$-\frac{397}{147511}$	$\frac{1+4X-4X^2-X^3}{5+X}$	0	$\frac{7\text{Log}[5]}{6}$
7	$\frac{369}{53471}$	$\frac{862}{695123}$	1	1	
11	$\frac{170}{11347}$	$-\frac{397}{147511}$	1	1	
13	$\frac{170}{11347}$	$-\frac{397}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{369}{53471}$	$\frac{862}{695123}$	1	1	
4861	$\frac{170}{11347}$	$-\frac{397}{147511}$	1	1	

Thus k_u($\frac{25}{13}$) = $-\frac{7\text{Log}[5]}{3}$

For u = $\frac{1724}{13}$*Lm1+ − $\frac{794}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{22}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{340}{11347}$	$-\frac{794}{147511}$	$\frac{1}{2}(1+X^3)$	1	
5	$\frac{340}{11347}$	$-\frac{794}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{738}{53471}$	$\frac{1724}{695123}$	1	1	
11	$\frac{340}{11347}$	$-\frac{794}{147511}$	$1+X$	2	
13	$\frac{340}{11347}$	$-\frac{794}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{738}{53471}$	$\frac{1724}{695123}$	1	1	
4861	$\frac{340}{11347}$	$-\frac{794}{147511}$	1	1	

Thus k_u($\frac{22}{13}$) = $-\frac{2\text{Log}[5]}{3}$

For u = $\frac{2586}{13}$*Lm1+ − $\frac{1191}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{17}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$\frac{1}{2}(1+X^2)$	1	
5	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{1107}{53471}$	$\frac{2586}{695123}$	1	1	
11	$\frac{510}{11347}$	$-\frac{1191}{147511}$	1	1	
13	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
17	$\frac{510}{11347}$	$-\frac{1191}{147511}$	$1+X$	2	
1621	$\frac{1107}{53471}$	$\frac{2586}{695123}$	1	1	
4861	$\frac{510}{11347}$	$-\frac{1191}{147511}$	1	1	

Thus k_u($\frac{17}{13}$) = $-\frac{2\text{Log}[5]}{3}$

For u = $\frac{3448}{13}$*Lm1+ $-\frac{1588}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{10}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$\frac{1}{2}(1-X^3)$	0	$\frac{3\text{Log}[2]}{2}$
5	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$\frac{1+4X+X^2}{5+X}$	1	
7	$\frac{1476}{53471}$	$\frac{3448}{695123}$	1	1	
11	$\frac{680}{11347}$	$-\frac{1588}{147511}$	1	1	
13	$\frac{680}{11347}$	$-\frac{1588}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{1476}{53471}$	$\frac{3448}{695123}$	1	1	
4861	$\frac{680}{11347}$	$-\frac{1588}{147511}$	1	1	

Thus k_u($\frac{10}{13}$) = $-3\text{Log}[2]$

For u = $\frac{4310}{13}$*Lm1+ $-\frac{1985}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{1}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{850}{11347}$	$-\frac{1985}{147511}$	$\frac{1}{2}(1+X^2)$	1	
5	$\frac{850}{11347}$	$-\frac{1985}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log}[5]}{6}$
7	$\frac{1845}{53471}$	$\frac{4310}{695123}$	1	1	
11	$\frac{850}{11347}$	$-\frac{1985}{147511}$	1	1	
13	$\frac{850}{11347}$	$-\frac{1985}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{1845}{53471}$	$\frac{4310}{695123}$	1	1	
4861	$\frac{850}{11347}$	$-\frac{1985}{147511}$	1	1	

Thus k_u($\frac{1}{13}$) = $-\frac{\text{Log}[5]}{3}$

For u = $\frac{6896}{13}$*Lm1+ $-\frac{3176}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{1}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	$\frac{1}{2}\left(1+X^2\right)$	1	
5	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{\text{Log[5]}}{6}$
7	$\frac{2952}{53471}$	$\frac{6896}{695123}$	1	1	
11	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	1	1	
13	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{2952}{53471}$	$\frac{6896}{695123}$	1	1	
4861	$\frac{1360}{11347}$	$-\frac{3176}{147511}$	1	1	

Thus k_u($\frac{1}{13}$) = $-\frac{\text{Log[5]}}{3}$

For u = $\frac{7758}{13}$*Lm1+ $-\frac{3573}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{10}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$\frac{1}{2}\left(1-X^3\right)$	0	$\frac{3\text{Log[2]}}{2}$
5	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$\frac{1+4X+X^2}{5+X}$	1	
7	$\frac{3321}{53471}$	$\frac{7758}{695123}$	1	1	
11	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	1	1	
13	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{3321}{53471}$	$\frac{7758}{695123}$	1	1	
4861	$\frac{1530}{11347}$	$-\frac{3573}{147511}$	1	1	

Thus k_u($\frac{10}{13}$) = -3Log[2]

For u = $\frac{8620}{13}$*Lm1+ $-\frac{3970}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{17}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$\frac{1}{2}(1+X^2)$	1	
5	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{Log[5]}{6}$
7	$\frac{3690}{53471}$	$\frac{8620}{695123}$	1	1	
11	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	1	1	
13	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
17	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	$1+X$	2	
1621	$\frac{3690}{53471}$	$\frac{8620}{695123}$	1	1	
4861	$\frac{1700}{11347}$	$-\frac{3970}{147511}$	1	1	

Thus k_u($\frac{17}{13}$) = $-\frac{2Log[5]}{3}$

For u = $\frac{9482}{13}$*Lm1+ $-\frac{4367}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{22}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$\frac{1}{2}(1+X^3)$	1	
5	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$\frac{1-X}{5+X}$	0	$\frac{Log[5]}{6}$
7	$\frac{369}{4861}$	$\frac{862}{63193}$	1	1	
11	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$1+X$	2	
13	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{369}{4861}$	$\frac{862}{63193}$	1	1	
4861	$\frac{1870}{11347}$	$-\frac{4367}{147511}$	1	1	

Thus k_u($\frac{22}{13}$) = $-\frac{2Log[5]}{3}$

For u = $\frac{10344}{13}$*Lm1+ $-\frac{4764}{13}$*Lm2 = u1*Lm1_p+u2*Lm2_p and m = $\frac{25}{13}$

p	u1	u2	W^*_{m,p}(s,u)	W^*_{m,p}(0,u)	d/ds W^*_{m,p}(0,u)
2	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$\frac{1}{2}(1+X^2)$	1	
5	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$\frac{1+4X-4X^2-X^3}{5+X}$	0	$\frac{7\text{Log}[5]}{6}$
7	$\frac{4428}{53471}$	$\frac{10344}{695123}$	1	1	
11	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	1	1	
13	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	
1621	$\frac{4428}{53471}$	$\frac{10344}{695123}$	1	1	
4861	$\frac{2040}{11347}$	$-\frac{4764}{147511}$	1	1	

Thus k_u($\frac{25}{13}$) = $-\frac{7\text{Log}[5]}{3}$

Summing these gives: kappa_0(2) = $-6\text{Log}[2]-8\text{Log}[5]$

To Recap:

kappa_0(3) = $-8\text{Log}[2]-4\text{Log}[3]-\frac{8\text{Log}[5]}{3}$

kappa_0(2) = $-6\text{Log}[2]-8\text{Log}[5]$

Ideal Class Number: 2

Field Discriminant: 52

Normalizing Constant: C2 = 4

Final Result:

$-\frac{1}{4}$*(3*kappa_0(3)+ $-$ 2*kappa_0(2))-1*Log[C2]= $\text{Log}[2]+3\text{Log}[3]-2\text{Log}[5]$

Bibliography

[1] M. Alsina and P. Bayer, *Quaternion Orders, Quadratic Forms, and Shimura Curves*, CRM Monograph Series **22** (2004).

[2] R. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998) 491-562.

[3] R. Borcherds, *Reflection Groups of Lorentzian Lattices*, Duke Mathematics Journal **104** (2000) no. 2, 319-366.

[4] J.E. Cremona, *Algorithms for Modular Elliptic Curves*, Cambridge University Press, 1992.

[5] N. Elkies, *Shimura Curve Computations*, Algorithmic Number Theory, Lecture Notes in Computer Science **1423** (1998) 1-47.

[6] B. Gross and D. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191-220.

[7] S. Johansson, *On Fundamental Domains of Arithmetic Fuchsian Groups*, Mathematics of Computation **69** (2000) no. 229, 339-349.

[8] S. S. Kudla, *Integrals on Borcherds Forms*, Compositio Math. **137** (2003), no. 3, 293-349.

[9] S. S. Kudla, *Special cycles and derivative of Eisenstein series*, Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 243-270.

[10] S. S. Kudla, M. Rapoport, and T. Yang, *On the derivative of an Eisenstein series of weight one*, Internat. Math. Res. Notices (1999), no. 7, 347-385.

[11] J. Schofer, *Borcherds Forms and Generalizations of Singular Moduli*, Ph.D. Thesis, University of Maryland, College Park, 2005.

[12] M.-F. Vignéras, *Arithmétique des Algèbres de Quaternions*, Lecture Notes in Mathematics **800** (1980).

[13] T. Yang, *preprint.*