Constructing Vector-Valued Modular Forms From Scalar Ones

Eric Errthum

Spotlight on Graduate Research Talk
Nov 2006
Run-of-the-Mill Modular Forms with Type

Let:

- $\mathfrak{h} = \{ z \in \mathbb{C} \mid \Re(z) > 0 \}$ be the upper half plane.
- $\text{SL}_2(\mathbb{Z})$ act on $\mathfrak{h}$ via linear fractional transformations, i.e.
  $$\gamma z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}$$
- $j(\gamma, z) = cz + d$

and choose

- A character (1-dim. rep.), $\chi : \text{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times$
- A subgroup, $\Gamma \subset \text{SL}_2(\mathbb{Z})$

**Example:** $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \right\}$

**Definition:** A function $f : \mathfrak{h} \to \mathbb{C}$ is a modular form on $\Gamma$ of weight $k \in \mathbb{Z}$ and type $\chi$ if it satifies for every $\gamma \in \Gamma$

$$f(\gamma z) = j(\gamma, z)^k \chi(\gamma)f(z).$$

**Examples:**

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad \text{where } q = e^{2\pi i z}$$

$$E_2(z, \chi_3) = \sum_{n \geq 1} q^n \sum_{d | n} d^2 \chi_3(n/d) \quad \text{where } \chi_3(\cdot) = \left( \frac{\cdot}{3} \right)$$
Slash Operator

Define

\[(f|_{k,\chi}\gamma)(z) = j(\gamma, z)^{-k}f(\gamma z)\]

Then

\[f \text{ Modular Form type } \chi, \text{ weight } k \iff f|_{k,\chi}\gamma = \chi(\gamma)f\]

for all \(\gamma \in \Gamma\)

Double Cover of \(SL_2(\mathbb{Z})\)

\[\tilde{SL}_2(\mathbb{Z}) = \left\{ (\gamma, \pm \sqrt{c\tau + d}) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\} \]

is a double cover of \(SL_2(\mathbb{Z})\).

\(\tilde{SL}_2(\mathbb{Z})\) still acts on \(\mathfrak{h}\) via linear fractional transformations.
Vector-Valued Modular Forms with Type

Choose

- A $n$-dim. rep., $\rho : \widetilde{\SL_2}(\mathbb{Z}) \to \GL_n(\mathbb{C})$
- A discrete subgroup, $\tilde{\Gamma} \subset \widetilde{\SL_2}(\mathbb{Z})$

**Definition:** A function $F : \mathfrak{k} \to \mathbb{C}^n$ is a modular form on $\tilde{\Gamma}$ of weight $k \in \frac{1}{2}\mathbb{Z}$ and type $\rho$ if it satisfies

$$F(\gamma z) = j(\gamma, z)^k \rho(\gamma) F(z)$$

for every $\gamma \in \tilde{\Gamma}$.

OR

$$F|_{k, \rho, \gamma} = \rho(\gamma) F \quad \text{for} \quad \gamma \in \tilde{\Gamma}$$

Compare to scalar-valued criterion:

$$f|_{k, \chi, \gamma} = \chi(\gamma) f \quad \text{for} \quad \gamma \in \Gamma$$

**Examples:**

For $n = 1$, $\tilde{\Gamma} = \overline{\Gamma_0(4)}$, $k = \frac{1}{2}$, and $\chi_\theta(\gamma) = \mp i^{(d-1)/2} \left( \frac{c}{d} \right)$

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$$

For $n > 1$: ?????????????
The $\mathbb{C}$-Vector Space

Let $\Lambda$ be a finite abelian (additive) group with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $Q_N$ such that $NQ_N(\lambda) = 0$ for all $\lambda \in \Lambda$.

**Example:** A lattice $L$ with quadratic form, $\Lambda = L^\vee / L$.

Take $\mathbb{C}^{\lvert \Lambda \rvert}$ with basis $\{e_\lambda\}_{\lambda \in \Lambda}$.

The Character and Representation

We’ll need a scalar-valued modular form of weight $k$ and character

$$\chi_\Lambda = \chi_\theta \cdot \left( \frac{d}{2^{2k} \lvert \Lambda \rvert} \right)^{2k+\left( \frac{1}{\lvert \Lambda \rvert} \right)^{-1}}$$

**Example:** A product of forms of the type $\Delta^{\frac{i}{2}}(iz)$ for $i, j \in \mathbb{Z}$.

Choose $\rho_\Lambda$ such that it satisfies

$$\rho_\Lambda(\gamma)e_\lambda = \chi_\Lambda(\gamma)e_{a\lambda}$$

for all $\gamma \in \widetilde{\Gamma_0(N)}$ and all $\lambda \in \Lambda$ with $Q_N(\lambda) = 0$.

**Example:** Weil Representation
The Construction

For \( f \) a modular form on \( \Gamma_0(N) \) of weight \( k \) and type \( \chi_\Lambda \), define

\[
F_f = \sum_{\gamma \in \Gamma_0(N) / \text{SL}_2(\mathbb{Z})} (f|_{k, \chi_\Lambda} \gamma) \rho_\Lambda(\gamma^{-1}) e_0.
\]

Then \( F_f \) is a \( \mathbb{C}^{|\Lambda|} \)-valued modular form on \( \Gamma_0(N) \) of weight \( k \) and type \( \rho_\Lambda \).

**Example:** \( \Lambda = \frac{1}{2} \mathbb{Z} / \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} / \mathbb{Z} \) with the quadratic form

\[
Q_4((a, b)) = a^2 + b^2.
\]

\[
f = \frac{\Delta_{24}^{10}(2z)}{\Delta_{24}^{4}(z) \Delta_{24}^{4}(4z)}
\]

is weight 1 with the correct character.

Let \( f_i = f|_{k, \chi_\Lambda} \gamma_i \). Then

\[
F_f = \begin{pmatrix}
    f_1 - \frac{i}{2}(f_2 - f_3 + f_5 + f_6) \\
    \frac{i}{2}(f_6 - f_2) - \frac{1}{2}(f_3 + f_5) \\
    \frac{i}{2}(f_6 - f_2) - \frac{1}{2}(f_3 + f_5) \\
    -f_4 + \frac{i}{2}(f_5 - f_2 - f_3 - f_6)
\end{pmatrix}
\]

is a \( \mathbb{C}^4 \)-valued modular form on \( \Gamma_0(4) \) of weight 1 and type \( \rho_\Lambda \).

**Extra Property:** \( F_f \) is invariant under transformations \( T \) that satisfy

\[
T(e_\lambda) = e_{\lambda'} \Rightarrow Q_N(\lambda) = Q_N(\lambda')
\]