p-Egyptian Fractions

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April 6th, 2011
Definition and Examples

Definition (Egyptian Fraction)

An **Egyptian Fraction** is a sum of distinct positive unit fractions that is less than one.

Ex:
Definition and Examples

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- \( \frac{5}{6} = \frac{1}{2} + \)
Definition (Egyptian Fraction)

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Ex:

- \( \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \)
Definition (Egyptian Fraction)

An **Egyptian Fraction** is a sum of distinct positive unit fractions that is less than one.

**Ex:**

- \( \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \)
- \( \frac{5}{121} = \frac{1}{25} \)
Definition and Examples

Definition (Egyptian Fraction)

An **Egyptian Fraction** is a sum of distinct positive unit fractions that is less than one.

Ex:

- $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$
- $\frac{5}{121} = \frac{1}{25} + \frac{1}{757}$
Definition and Examples

Definition (Egyptian Fraction)

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- \( \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \)
- \( \frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} \)
Definition and Examples

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- \( \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \)
- \( \frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180912} \)
Definition and Examples

**Definition (Egyptian Fraction)**

An **Egyptian Fraction** is a sum of distinct positive unit fractions that is less than one.

**Ex:**

\[
\frac{5}{6} = \frac{1}{2} + \frac{1}{3}
\]

\[
\frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180912} + \frac{1}{1527612795642093418846225}
\]
Definition and Examples

Definition (Egyptian Fraction)

An **Egyptian Fraction** is a sum of distinct positive unit fractions that is less than one.

Ex:

- \( \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \)
- \( \frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180912} + \frac{1}{1527612795642093418846225} \)
- \( \frac{5}{121} = \frac{1}{45} + \frac{1}{75} + \frac{1}{300} + \frac{1}{1023} + \frac{1}{1089} + \frac{1}{1860} \)
Definition and Examples

Definition (Egyptian Fraction)

An **Egyptian Fraction** is a sum of distinct positive unit fractions that is less than one.

**Ex:**

- \[ \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \]
- \[ \frac{5}{121} = \frac{1}{25} + \frac{1}{75} + \frac{1}{763309} + \frac{1}{873960180912} + \frac{1}{1527612795642093418846225} \]
- \[ \frac{5}{121} = \frac{1}{45} + \frac{1}{75} + \frac{1}{300} + \frac{1}{1023} + \frac{1}{1089} + \frac{1}{1860} \]
- \[ \frac{5}{121} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363} \]
Basic Facts

Fact (Not Unique)

There is more than one way to expand a rational number into an Egyptian fraction.
Basic Facts

Fact (Not Unique)

There is more than one way to expand a rational number into an Egyptian fraction.

Fact (Existence)

For all positive rational numbers less than one there exists an Egyptian fraction expansion.
Finding an Egyptian fraction

Greedy Method

Example

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Example

Find the largest unit fraction less than $4/13$. 
Greedy Method

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Find the largest unit fraction less than $\frac{4}{13}$.

$$\frac{4}{13} - \frac{1}{q} = \frac{r}{13q}$$
Finding an Egyptian fraction

**Greedy Method**

**Example**

Find the largest unit fraction less than $4/13$.

\[
\frac{4}{13} - \frac{1}{q} = \frac{r}{13q} \\
\frac{4q - 13}{13q} = \frac{r}{13q}
\]
Finding an Egyptian fraction

Greedy Method

Example

Find the largest unit fraction less than $4/13$.

\[
\frac{4}{13} - \frac{1}{q} = \frac{r}{13q} \\
\frac{4q - 13}{13q} = \frac{r}{13q} \\
4q - 13 = r
\]
Finding an Egyptian fraction

Greedy Method

Example

Find the largest unit fraction less than $\frac{4}{13}$.

\[
\frac{4}{13} - \frac{1}{q} = \frac{r}{13q}
\]

\[
\frac{4q - 13}{13q} = \frac{r}{13q}
\]

\[
4q - 13 = r
\]

\[
13 = 4q - r
\]
Greedy Method

The **division algorithm** gives the largest $q'$ such that $r'$ remains positive.

$$13 = 4q' + r' \quad q' = 3 \quad r' = 1$$

$r$ smaller than 4
Finding an Egyptian fraction

Greedy Method

The **division algorithm** gives the largest $q'$ such that $r'$ remains positive.

$$13 = 4q' + r'$$

$q' = 3$ $r' = 1$

Increment $q'$ by 1 to get the smallest $q$ such that $r$ is positive.

$$13 = 4q - r$$

$q = 4$ $r = 3$

$r$ smaller than 4
Greedy Method

The **division algorithm** gives the largest $q'$ such that $r'$ remains positive.

$$13 = 4q' + r'$$

$q' = 3, r' = 1$

Increment $q'$ by 1 to get the smallest $q$ such that $r$ is positive.

$$13 = 4q - r$$

$q = 4, r = 3$

$r$ smaller than 4 Then $q = 4$ is as small as it can be and produces the largest possible unit fraction $1/4$ that is less than $4/13$. 

Acknowledgements
Finding an Egyptian fraction

Greedy Method

Example

\[ \frac{4}{13} = \frac{1}{4} + \frac{3}{52} \quad \text{Not all unit fractions!} \]
Finding an Egyptian fraction

**Greedy Method**

Example

\[
\frac{4}{13} = \frac{1}{4} + \frac{3}{52} \quad \text{Not all unit fractions!}
\]

\[
52 = 3(18) - 1
\]
Finding an Egyptian fraction

**Greedy Method**

**Example**

\[
\frac{4}{13} = \frac{1}{4} + \frac{3}{52} \quad \text{Not all unit fractions!}
\]

\[
\frac{3}{52} = \frac{1}{18} + \frac{1}{468} \quad \text{Done!}
\]

\[
52 = 3(18) - 1
\]
Finding an Egyptian fraction

Greedy Method

**Example**

\[
\frac{4}{13} = \frac{1}{4} + \frac{3}{52} \quad \text{Not all unit fractions!}
\]

\[
\frac{3}{52} = \frac{1}{18} + \frac{1}{468} \quad \text{Done!}
\]

\[
\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}
\]

\[
52 = 3(18) - 1
\]
Greedy Method

Choose a fraction \( \frac{a}{b} < 1 \)

\[
\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1}
\]
Finding an Egyptian fraction

Greedy Method

Choose a fraction \( \frac{a}{b} < 1 \)

\[
\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1}
\]

\( b = aq_1 - r_1 \)
Finding an Egyptian fraction

**Greedy Method**

Choose a fraction \( \frac{a}{b} < 1 \)

\[
\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1} \quad \quad b = aq_1 - r_1
\]

Repeat on \( \frac{r_1}{bq_1} \)

\[
\frac{r_1}{bq_1} - \frac{1}{q_2} = \frac{r_2}{bq_1 q_2}
\]
Finding an Egyptian fraction

Greedy Method

Choose a fraction $\frac{a}{b} < 1$

$$\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1}$$

Repeat on $\frac{r_1}{bq_1}$

$$\frac{r_1}{bq_1} - \frac{1}{q_2} = \frac{r_2}{bq_1q_2}$$

$$b = aq_1 - r_1$$

$$bq_1 = r_1q_2 - r_2$$
Finding an Egyptian fraction

Greedy Method

Choose a fraction \( \frac{a}{b} < 1 \)

\[
\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1} \\
\]

Repeat on \( \frac{r_1}{bq_1} \)

\[
\frac{r_1}{bq_1} - \frac{1}{q_2} = \frac{r_2}{bq_1q_2} \\
\]

\[
b = aq_1 - r_1 \\
bq_1 = r_1q_2 - r_2 \\
bq_1q_2 = r_2q_3 - r_3 \\
\]
Finding an Egyptian fraction

Greedy Method

Choose a fraction \( \frac{a}{b} < 1 \)

\[
\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1}
\]

Repeat on \( \frac{r_1}{bq_1} \)

\[
\frac{r_1}{bq_1} - \frac{1}{q_2} = \frac{r_2}{bq_1q_2}
\]

... 

\[
b = aq_1 - r_1
\]

\[
bq_1 = r_1q_2 - r_2
\]

\[
bq_1q_2 = r_2q_3 - r_3
\]

... 

\[
bq_1q_2 \cdots q_{n-1} = r_{n-1}q_n - 0
\]
Finding an Egyptian fraction

Greedy Method

Choose a fraction \( \frac{a}{b} < 1 \)

\[
\frac{a}{b} - \frac{1}{q_1} = \frac{r_1}{bq_1}
\]

Repeat on \( \frac{r_1}{bq_1} \)

\[
\frac{r_1}{bq_1} - \frac{1}{q_2} = \frac{r_2}{bq_1q_2}
\]

\[
b = aq_1 - r_1
\]

\[
bq_1 = r_1q_2 - r_2
\]

\[
bq_1q_2 = r_2q_3 - r_3
\]

\[
\vdots
\]

\[
bq_1q_2 \cdots q_{n-1} = r_{n-1}q_n - 0
\]

\[
\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} = \frac{a}{b}
\]
p-Adic Size

A number is small if it is very divisible by $p$. 
p-Adic Size

A number is small if it is very divisible by $p$.

**Definition (p-Adic Size)**

Let $n$ be the power of $p$ in the factorization of $\frac{a}{b}$ and take $|\frac{a}{b}|_p$ to be $p^{-n}$. 
A number is small if it is very divisible by $p$.

**Definition (p-Adic Size)**

Let $n$ be the power of $p$ in the factorization of $\frac{a}{b}$ and take $|\frac{a}{b}|_p$ to be $p^{-n}$.

\[
|294|_7 = |7^2 \cdot 3 \cdot 2|_7 = |7^2|_7 = \frac{1}{49} \quad \text{SMALL}
\]
A number is small if it is very divisible by $p$.

**Definition (p-Adic Size)**

Let $n$ be the power of $p$ in the factorization of $\frac{a}{b}$ and take $|\frac{a}{b}|_p$ to be $p^{-n}$.

\[
|294|_7 = |7^2 \cdot 3 \cdot 2|_7 = |7^2|_7 = \frac{1}{49} \text{ SMALL}
\]
\[
\left| \frac{30}{189} \right|_3 = \left| \frac{2 \cdot 3 \cdot 5}{7 \cdot 3^3} \right|_3 = \left| \frac{3}{3^3} \right|_3 = \left| \frac{1}{3^2} \right|_3 = 9 \text{ LARGE}
\]
p-Adic Size

A number is small if it is very divisible by \( p \).

**Definition (p-Adic Size)**

Let \( n \) be the power of \( p \) in the factorization of \( \frac{a}{b} \) and take \( \left| \frac{a}{b} \right|_p \) to be \( p^{-n} \).

\[
|294|_7 = |7^2 \cdot 3 \cdot 2|_7 = |7^2|_7 = \frac{1}{49} \quad \text{SMALL}
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\[
\left| \frac{30}{189} \right|_3 = \left| \frac{2 \cdot 3 \cdot 5}{7 \cdot 3^3} \right|_3 = \left| \frac{3}{3^3} \right|_3 = \left| \frac{1}{3^2} \right|_3 = 9 \quad \text{LARGE}
\]

\[
\left| \frac{a}{b} \right|_p \quad a \text{ and } b \text{ contain the same power of } p, \quad \left| \frac{a}{b} \right|_p = 1 \quad \text{SAME}
\]
p-Adic Division Algorithm

**Definition (Errthum, Lager, 2009)**

Let $p$ be an odd prime. Then given any $b$ and $a \in \mathbb{Q}_p$ where $a \neq 0$, there exists uniquely $q' = \frac{m}{p^k}$ with $|q'| < p$, and $r' \in \mathbb{Q}_p$ with $|r'|_p < |a|_p$ such that $b = aq' + r'$. 

"..."
p-Adic Division Algorithm

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Let $p$ be an odd prime. Then given any $b$ and $a \in \mathbb{Q}_p$ where $a \neq 0$, there exists uniquely $q' = \frac{m}{p^k}$ with $|q'| < p$, and $r' \in \mathbb{Q}_p$ with $|r'| < |a|_p$ such that $b = aq' + r'$.

$\frac{5}{6}$ divided by $-\frac{2}{3}$ is 4 with a remainder of $\frac{7}{2}$ in the 7-adics.

$$\frac{5}{6} = \left( -\frac{2}{3} \right) 4 + \frac{7}{2}$$
p-Adic Division Algorithm

Definition (Errthum, Lager, 2009)

Let $p$ be an odd prime. Then given any $b$ and $a \in \mathbb{Q}_p$ where $a \neq 0$, there exists uniquely $q' = \frac{m}{p^k}$ with $|q'|_p < p$, and $r' \in \mathbb{Q}_p$ with $|r'|_p < |a|_p$ such that $b = aq' + r'$.

\[
\frac{5}{6} \text{ divided by } -\frac{2}{3} \text{ is 4 with a remainder of } \frac{7}{2} \text{ in the 7-adics.}
\]

\[
20 \text{ divided by 5 is 4 with a remainder of 0 in the 7-adics.}
\]

\[
\frac{5}{6} = \left(-\frac{2}{3}\right) 4 + \frac{7}{2}
\]

\[
20 = 5 \cdot 4 + 0
\]
**p-Adic Division Algorithm**

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Let $p$ be an odd prime. Then given any $b$ and $a \in \mathbb{Q}_p$ where $a \neq 0$, there exists uniquely $q' = \frac{m}{p^k}$ with $|q'|_p < p$, and $r' \in \mathbb{Q}_p$ with $|r'|_p < |a|_p$ such that $b = aq' + r'$.

\[
\frac{5}{6} \text{ divided by } -\frac{2}{3} \text{ is 4 with a remainder of } \frac{7}{2} \text{ in the 7-adics.}
\]

\[
20 \text{ divided by 5 is 4 with a remainder of 0 in the 7-adics.}
\]

\[
20 \text{ divided by 5 is 1 with a remainder of 15 in the 3-adics.}
\]

\[
\frac{5}{6} = \left(-\frac{2}{3}\right) 4 + \frac{7}{2}
\]

\[
20 = 5 \cdot 4 + 0 \\
20 = 5 \cdot 1 + 15
\]
p-Adic Greedy Algorithm

Classical

Choose $|\frac{a}{b}| \leq 1$.  

p-Adic

Choose $|\frac{a}{b}|_p \leq 1$.  

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# p-Adic Greedy Algorithm

## Classical

Choose $|\frac{a}{b}| \leq 1$.

Division algorithm

$$b, a \in \mathbb{Z} \rightarrow q', r' \in \mathbb{Z} \text{ with } 0 \leq r' < |a| \text{ and } b = aq' + r'$$

## p-Adic

Choose $|\frac{a}{b}|_p \leq 1$.

Division algorithm

$$b, a \in \mathbb{Q} \rightarrow q' = \frac{m}{p^k}, r' \in \mathbb{Q} \text{ with } 0 \leq |r'|_p < |a|_p \text{ and } b = aq' + r'$$
p-Adic Greedy Algorithm

Classical

Choose $|\frac{a}{b}| \leq 1$.

Division algorithm

$b, a \in \mathbb{Z} \rightarrow q', r' \in \mathbb{Z}$ with

$0 \leq r' < |a|$ and $b = aq' + r'$

Take $q = q' + 1$ so that $b = aq - r$.

p-Adic

Choose $|\frac{a}{b}|_p \leq 1$.

Division algorithm

$b, a \in \mathbb{Q} \rightarrow q' = \frac{m}{p^k}, r' \in \mathbb{Q}$ with

$0 \leq |r'|_p < |a|_p$ and $b = aq' + r'$

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p-Adic Greedy Algorithm

Classical

Choose $\left| \frac{a}{b} \right| \leq 1$.

Division algorithm
$b, a \in \mathbb{Z} \rightarrow q', r' \in \mathbb{Z}$ with
$0 \leq r' < |a|$ and $b = aq' + r'$

Take $q = q' + 1$ so that $b = aq - r$.

p-Adic

Choose $\left| \frac{a}{b} \right|_p \leq 1$.

Division algorithm
$b, a \in \mathbb{Q} \rightarrow q' = \frac{m}{p^k}, r' \in \mathbb{Q}$ with
$0 \leq |r'|_p < |a|_p$ and $b = aq' + r'$

Take $q = q' + ?$ so that $b = aq - r$. 
What is?

In the classical case,

\[ q = q' + 1 \]
What is?

In the classical case,

\[ q = q' + 1 \]

In the \( p \)-adic case,

\[ q = q' + \left\lfloor \frac{b}{a} - q' \right\rfloor p \]
What is ?

In the classical case,

\[ q = q' + 1 \]

In the p-adic case,

\[ q = q' + \left\lfloor \frac{b}{a} - q' \right\rfloor p \]

Choose \( p = 1 \) and pretend it’s the classical case,

\[ q = q' + \left\lfloor \frac{b}{a} - \left\lfloor \frac{b}{a} \right\rfloor \right\rfloor 1 \]

Just equal to 1
p-Adic Greedy Algorithm

Classical

Choose \( \left| \frac{a}{b} \right| \leq 1. \)

Division algorithm

\[
\begin{align*}
&b, a \in \mathbb{Z} \longrightarrow q', r' \in \mathbb{Z} \text{ with} \\
&0 \leq r' < |a| \text{ and } b = aq' + r'
\end{align*}
\]

Take \( q = q' + 1 \) so that \( b = aq - r. \)

p-Adic

Choose \( \left| \frac{a}{b} \right|_p \leq 1. \)

Division algorithm

\[
\begin{align*}
&b, a \in \mathbb{Q} \longrightarrow q' = \frac{m}{p^k}, r' \in \mathbb{Q} \text{ with} \\
&0 \leq |r'|_p < |a|_p \text{ and } b = aq' + r'
\end{align*}
\]

Take \( q = q' + \left\lceil \frac{b}{a} \right\rceil_p \) so that \( b = aq - r. \)
p-Adic Greedy Algorithm

**Classical**

Choose $\left| \frac{a}{b} \right| \leq 1$.

Division algorithm

$\begin{align*}
    b, a &\in \mathbb{Z} \rightarrow q', r' \in \mathbb{Z} \\
    0 &\leq r' < |a| \text{ and } b = aq' + r'
\end{align*}$

Take $q = q' + 1$ so that $b = aq - r$.

**p-Adic**

Choose $\left| \frac{a}{b} \right|_p \leq 1$.

Division algorithm

$\begin{align*}
    b, a &\in \mathbb{Q} \rightarrow q' = \frac{m}{p^k}, r' \in \mathbb{Q} \\
    0 &\leq |r'|_p < |a|_p \text{ and } b = aq' + r'
\end{align*}$

Take $q = q' + \left\lfloor \frac{b - q'}{p} \right\rfloor p$ so that $b = aq - r$. 
Different kind of Egyptian...

**p-Adic Egyptian Fraction**

Instead of fractions of the form,

$$\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}$$

with distinct terms, allow for increasing whole powers of a prime $p$ to appear in the numerator,

$$\frac{p^{\varepsilon_1}}{d_1} + \frac{p^{\varepsilon_2}}{d_2} + \frac{p^{\varepsilon_3}}{d_3} + \cdots + \frac{p^{\varepsilon_n}}{d_n}$$

with $0 \leq \varepsilon_n < \varepsilon_{n+1}$. 
Numerical Example

Find the $p$-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$. 
Numerical Example

Find the p-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$.

\[
b = aq' + r' \quad q = q' + \left\lfloor \frac{b - q'}{p} \right\rfloor p \quad b = aq - r
\]
Numerical Example

Find the $p$-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$.

\[
b = aq' + r' \quad q = q' + \left\lceil \frac{b - q'}{p} \right\rceil p \quad b = aq - r
\]

\[
9 = 2 \cdot 2 + 5
\]
Numerical Example

Find the p-Egyptian fraction expansion of \( \frac{2}{9} \) in \( \mathbb{Q}_5 \).

\[
b = aq' + r' \quad q = q' + \left\lceil \frac{b - q'}{p} \right\rceil p \quad b = aq - r
\]

\[
9 = 2 \cdot 2 + 5 \quad q_1 = 2 + \left\lceil \frac{9 - 2}{5} \right\rceil 5
\]
Numerical Example

Find the p-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$.

$$b = aq' + r'$$
$$q = q' + \left\lceil \frac{b - q'}{p} \right\rceil p$$

$$b = aq - r$$

$9 = 2 \cdot 2 + 5$

$q_1 = 2 + \left\lceil \frac{9 - 2}{5} \right\rceil 5$

$= 2 + \left\lceil \frac{1}{2} \right\rceil 5 = 7$
Numerical Example

Find the p-Egyptian fraction expansion of \( \frac{2}{9} \) in \( \mathbb{Q}_5 \).

\[
b = aq' + r'
\]
\[
q = q' + \left\lfloor \frac{b - q'}{p} \right\rfloor p
\]
\[
b = aq - r
\]

\[
9 = 2 \cdot 2 + 5
\]
\[
q_1 = 2 + \left\lfloor \frac{9 - 2}{5} \right\rfloor 5
\]
\[
= 2 + \left\lfloor \frac{1}{2} \right\rfloor 5 = 7
\]
\[
9 = 2 \cdot 7 - 5
\]
Numerical Example

Find the p-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$.

\[ b = aq' + r' \quad q = q' + \left\lfloor \frac{b - q'}{p} \right\rfloor p \quad b = aq - r \]

\[ 9 = 2 \cdot 2 + 5 \quad q_1 = 2 + \left\lfloor \frac{9 - 2}{5} \right\rfloor 5 \]

\[ = 2 + \left\lfloor \frac{1}{2} \right\rfloor 5 = 7 \quad 9 = 2 \cdot 7 - 5 \]

\[ 9 \cdot 7 = 5 \left( \frac{13}{5} \right) + 50 \]
Numerical Example

Find the p-Egyptian fraction expansion of \( \frac{2}{9} \) in \( \mathbb{Q}_5 \).

\[
b = aq' + r' \\
q = q' + \left\lfloor \frac{b-q'}{p} \right\rfloor p \\
b = aq - r
\]

\[
9 = 2 \cdot 2 + 5 \\
q_1 = 2 + \left\lfloor \frac{9-2}{5} \right\rfloor 5 \\
= 2 + \left\lfloor \frac{1}{2} \right\rfloor 5 = 7 \\
9 = 2 \cdot 7 - 5
\]

\[
9 \cdot 7 = 5 \left( \frac{13}{5} \right) + 50 \\
q_2 = \frac{13}{5} + \left\lfloor \frac{63}{5} \frac{13}{5} \right\rfloor 5
\]
Numerical Example

Find the p-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$.

\[
\begin{align*}
   b &= aq' + r' \\
   q &= q' + \left\lfloor \frac{b - q'}{p} \right\rfloor p \\
   b &= aq - r
\end{align*}
\]

\[
\begin{align*}
   9 &= 2 \cdot 2 + 5 \\
   q_1 &= 2 + \left\lfloor \frac{9 - 2}{5} \right\rfloor 5 \\
      &= 2 + \left\lfloor \frac{1}{2} \right\rfloor 5 = 7 \\
   9 &= 2 \cdot 7 - 5
\end{align*}
\]

\[
\begin{align*}
   9 \cdot 7 &= 5 \left(\frac{13}{5}\right) + 50 \\
   q_2 &= \frac{13}{5} + \left\lfloor \frac{63}{5} \frac{13}{5} \right\rfloor 5 \\
      &= \frac{13}{5} + \left\lfloor 2 \right\rfloor 5 = \frac{63}{5}
\end{align*}
\]
Numerical Example

Find the p-Egyptian fraction expansion of \( \frac{2}{9} \) in \( \mathbb{Q}_5 \).

\[
b = aq' + r' \quad q = q' + \left\lceil \frac{b - q'}{p} \right\rceil p \quad b = aq - r
\]

\[
9 = 2 \cdot 2 + 5 \quad q_1 = 2 + \left\lceil \frac{9 - 2}{5} \right\rceil 5
\]

\[
= 2 + \left\lceil \frac{1}{2} \right\rceil 5 = 7 \quad 9 = 2 \cdot 7 - 5
\]

\[
9 \cdot 7 = 5 \left( \frac{13}{5} \right) + 50 \quad q_2 = \frac{13}{5} + \left\lceil \frac{63}{5} \right\rceil \frac{13}{5} \cdot 5
\]

\[
= \frac{13}{5} + \left\lceil 2 \right\rceil 5 = \frac{63}{5} \quad 63 = 5 \left( \frac{63}{5} \right) - 0
\]
Numerical Example

Find the $p$-Egyptian fraction expansion of $\frac{2}{9}$ in $\mathbb{Q}_5$.

\[
\begin{align*}
  b &= aq' + r' \\
  q &= q' + \left\lfloor \frac{b - q'}{p} \right\rfloor p \\
  b &= aq - r
\end{align*}
\]

\[
\begin{align*}
  9 &= 2 \cdot 2 + 5 \\
  q_1 &= 2 + \left\lfloor \frac{9 - 2}{5} \right\rfloor 5 \\
  &= 2 + \left\lfloor \frac{1}{2} \right\rfloor 5 = 7 \\
  9 &= 2 \cdot 7 - 5 \\
  9 \cdot 7 &= 5 \left( \frac{13}{5} \right) + 50 \\
  q_2 &= \frac{13}{5} + \left\lfloor \frac{63}{5} \cdot \frac{13}{5} \right\rfloor 5 \\
  &= \frac{13}{5} + \left\lfloor 2 \right\rfloor 5 = \frac{63}{5} \\
  63 &= 5 \left( \frac{63}{5} \right) - 0 \\
  \frac{1}{7} + \frac{5}{63} &= \frac{2}{9}
\end{align*}
\]
Alternate Egyptian fraction expansion

\[ \frac{a}{b} = \frac{p^{\varepsilon_1}}{d_1} + \frac{p^{\varepsilon_2}}{d_2} + \cdots + \frac{p^{\varepsilon_n}}{d_n} \]
Alternate Egyptian fraction expansion

\[ \frac{a}{b} = \frac{p^{\varepsilon_1}}{d_1} + \frac{p^{\varepsilon_2}}{d_2} + \cdots + \frac{p^{\varepsilon_n}}{d_n} \]

\[ \frac{a}{b \cdot p^{\varepsilon_n}} = \frac{1}{d_1 \cdot p^{\varepsilon_n - \varepsilon_1}} + \frac{1}{d_2 \cdot p^{\varepsilon_n - \varepsilon_2}} + \cdots + \frac{1}{d_n} \]
Alternate Egyptian fraction expansion

\[
\frac{a}{b} = \frac{p^{\varepsilon_1}}{d_1} + \frac{p^{\varepsilon_2}}{d_2} + \cdots + \frac{p^{\varepsilon_n}}{d_n}
\]

\[
\frac{a}{b \cdot p^{\varepsilon_n}} = \frac{1}{d_1 \cdot p^{\varepsilon_{n-1}}} + \frac{1}{d_2 \cdot p^{\varepsilon_{n-2}}} + \cdots + \frac{1}{d_n}
\]

How do we make \(\frac{a}{b \cdot p^{\varepsilon_n}}\) the input?
Acknowledgements

- Professor Errthum
- Winona State University, Foundation