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CONTINUED FRACTIONS IN LOCAL FIELDS, II

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ABSTRACT. The present paper is a continuation of an earlier work by the author. We propose some new definitions of p-adic continued fractions. At the end of the paper we give numerical examples illustrating these definitions. It turns out that for every m, $1 < m < 5000$, $5 \nmid m$ if $\sqrt{m} \in \mathbb{Q}_5 \setminus \mathbb{Q}$, then $\sqrt{m}$ has a periodic continued fraction expansion. The same is not true in $\mathbb{Q}_p$ for some larger values of p.

1. INTRODUCTION

There are two nonequivalent definitions of continued fractions in the field $\mathbb{Q}_p$ of p-adic numbers: the definition of T. Schneider ([Sch], 1968) and the definition of A.A. Ruban ([Ru], 1970) modified by the author ([Br], 1978) and rediscovered by L.X. Wang ([Wa1], 1985).

In the first definition we consider continued fractions of the form (we assume for simplicity that $p > 2$, and $v(\alpha) = 0$, where v is the p-adic valuation, and $\alpha \in \mathbb{Q}_p$)

$$\alpha = b_0 + \cfrac{a_0}{b_1 + \cfrac{a_1}{b_2 + \cdots}},$$

where $b_j \in \{1, 2, \ldots, p-1\}$, $a_j = p^{\alpha_j}, \alpha_j \geq 1$, for $j \geq 0$.

In the second definition we assume that $a_j = 1$ for $j \geq 0$, and

$$b_j \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p) \quad \text{in [Ru]}, \qquad \text{and} \qquad b_j \in \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right) \quad \text{in [Br]}.$$

Moreover, $v(b_0) = 0$ and $v(b_j) < 0$ for $j \geq 1$. The precise definition is given in Algorithm I below. The following natural elementary questions arise:

1. Can every $\alpha \in \mathbb{Q}_p$ be written uniquely as a finite or infinite continued fraction?
2. Can every rational number $\alpha \in \mathbb{Q}$ be written as a finite continued fraction? If not, determine those infinite continued fractions which correspond to rational numbers.
3. Can every $\alpha \in \mathbb{Q}_p$ quadratic over $\mathbb{Q}$ be written as a periodic continued fraction?

The answers are as follows:

1. Yes, it follows easily from both definitions.

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2. No, for the definitions in [Sch] and [Ru]; yes, for the definition in [Br]. Infinite continued fractions corresponding to rational numbers have been described in [Bu] for Schneider continued fractions, and in [La] and [Wa1] for Ruban ones.
3. No, in general. For some particular α's the continued fraction is periodic.

There are several papers devoted to periodicity, e.g., [Beck], [Be1], [Be2], [Be3], [Be4], [Br], [Bu], [Dea], [La], [Ti], [Wa1], [We1], [We2].

B.M.M. de Weger [We1] has modified question 3 as follows. To every $\alpha \in \mathbb{Q}_p$, he attached a sequence of approximation lattices, and he proved that $\alpha \in \mathbb{Q}_p$ is quadratic over $\mathbb{Q}$ iff the corresponding sequence of approximation lattices is periodic. He remarked (see [We1], p.70), "... it seems that a simple and satisfactory p-adic continued fraction algorithm does not exist."

In the present paper we propose some new definitions of p-adic continued fractions. It seems that they are more satisfactory than earlier ones, since for many $\alpha \in \mathbb{Q}_p$, quadratic over $\mathbb{Q}$, the continued fraction in the new sense is periodic. See the numerical examples in Section 4.

2. Definitions

First we recall the definition of A.A. Ruban [Ru] and its modification [Br]. We assume for simplicity that p is an odd prime number. Let $\mathcal{R} \subset \mathbb{Q}$ be a set of representatives modulo p such that $0 \in \mathcal{R}$. Then every $\alpha \in \mathbb{Q}_p$ can be written uniquely in the form

$$\alpha = \sum_{n=r}^{\infty} a_n p^n, \tag{1}$$

where $r \in \mathbb{Z}$, $a_n \in \mathcal{R}$, for $n \geq r$ and $a_r \neq 0$ if $\alpha \neq 0$. Then $v(\alpha) = r$, where v is the p-adic valuation.

We define the mapping (integral part) $s : \mathbb{Q}_p \longrightarrow \mathbb{Q}$ as follows. For α given by (1) let

$$s(\alpha) = \sum_{n=r}^{0} a_n p^n.$$

In particular we can take $\mathcal{R} = \{0, 1, \dots, p-1\}$ (see [Ru]) or $\mathcal{R} = \left\{-\frac{p-1}{2}, \dots, -1, 0, 1, \dots, \frac{p-1}{2}\right\}$ (see [Br]). In the present paper we choose the second possibility. Then $s(\mathbb{Q}_p) = \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right)$.

Now we can proceed as in the classical definition of continued fractions in $\mathbb{R}$.

Algorithm I.

For a given $\alpha \in \mathbb{Q}_p$, we define inductively (finite or infinite) sequences (a_n) and (b_n) as follows.

Let $a_0 = \alpha$ and $b_0 = s(\alpha)$.

If $b_0 = a_0$, then a_1 and b_1 are not defined.

If $b_0 \neq a_0$, then let $a_1 = (a_0 - b_0)^{-1}$ and $b_1 = s(a_1)$.

If a_j, b_j are defined for $j = 0, 1, \dots, k$ and $b_k = a_k$, then a_{k+1} and b_{k+1} are not defined.

If $b_k \neq a_k$, then let $a_{k+1} = (a_k - b_k)^{-1}$ and $b_{k+1} = s(a_{k+1})$.

We call the sequence (b_n) the p-adic continued fraction of α.

From the definition of the mapping s it follows that $v(\beta - s(\beta)) > 0$ for every $\beta \in \mathbb{Q}_p$, and hence $v(b_n) < 0$ for $n \geq 1$.

For an arbitrary sequence (b_n), where $b_n \in s(\mathbb{Q}_p)$, and $v(b_n) < 0$, for $n \geq 1$, we define the partial quotients

$$\frac{A_n}{B_n} = [b_0; b_1, \dots, b_n] \qquad (n = 0, 1, \dots)$$

as usual:

$$A_0 = b_0,\ A_1 = b_0 b_1 + 1,\ A_n = b_n A_{n-1} + A_{n-2},\ \text{for } n \geq 2,$$

$$B_0 = 1, \quad B_1 = b_1, \qquad B_n = b_n B_{n-1} + B_{n-2},\ \text{for } n \geq 2.$$

If the sequence (b_n) is infinite, then the sequence (A_n/B_n) is convergent, and moreover if (b_n) was obtained by the above algorithm applied to $\alpha \in \mathbb{Q}_p$, then $\lim(A_n/B_n) = \alpha$. In this case we use the notation $\alpha = [b_0; b_1, \dots]$.

For the proofs see [Br].

3. Some new algorithms

Now we present some new definitions of p-adic continued fractions.

Lemma 1. *Let an infinite sequence (b_n) satisfy*

$$(3) \qquad \begin{cases} b_n \in \mathbb{Z}[\frac{1}{p}], & \text{for } n \geq 0,\\ v(b_{2n}) = 0, & \text{for } n > 0,\\ v(b_{2n+1}) < 0, & \text{for } n \geq 0, \end{cases}$$

and let $\frac{A_n}{B_n}$ be the n-th partial quotient corresponding to the sequence (b_n).

Then the sequence $\frac{A_n}{B_n}$ is convergent to a p-adic number α. We have $v(\alpha) \geq 0$ provided $v(b_0) \geq 0$.

Proof. 1) We prove by induction that $B_n \neq 0$ and $v(B_n) \leq v(B_{n-1})$, for every $n \geq 0$, and that the equality holds iff n is even.

Since $B_0 = 1$, $B_1 = b_1$, then $v(B_0) = 0 > v(b_1) = v(B_1)$, by the assumption. Consequently $B_1 \neq 0$.

Suppose that

$$(4) \qquad B_{n-1} \neq 0 \quad \text{and} \quad v(B_{n-1}) \leq v(B_{n-2}) \qquad \text{for some } n \geq 2$$

and that the equality holds iff n is odd.

In view of (3) and (4) we have

$$v(b_n B_{n-1}) = v(B_{n-1}) < v(B_{n-2}) \quad \text{for } n \text{ even}$$

and

$$v(b_n B_{n-1}) < v(B_{n-1}) = v(B_{n-2}) \quad \text{for } n \text{ odd}.$$

Thus in both cases $v(b_n B_{n-1}) < v(B_{n-2})$, and hence in view of $B_n = b_n B_{n-1} + B_{n-2}$ we get

$$\begin{aligned} v(B_n) &= \min(v(b_n B_{n-1}), v(B_{n-2})) \\ &= v(b_n B_{n-1}) \quad \begin{cases} = v(B_{n-1}) & \text{for } n \text{ even},\\ < v(B_{n-1}) & \text{for } n \text{ odd}. \end{cases} \end{aligned}$$

Consequently $v(B_n) < 0$, hence $B_n \neq 0$.

By induction the claim follows.

2) From the first part of the proof we get $\lim B_n^{-1} = 0$. The well-known formula

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{(-1)^n}{B_{n+1}B_n}$$

implies that $\left(\frac{A_n}{B_n}\right)$ is a Cauchy sequence. Hence it is convergent in $\mathbb{Q}_p$. □

We shall prove that every p-adic number α can be written in the form $\alpha = [b_0; b_1, \dots]$ for some finite or infinite sequence (b_n) satisfying (3). Of course such a representation is not unique. To get the uniqueness it is necessary to make some further restrictions on the sequence (b_n), e.g., such as stated in the algorithms below.

For $\alpha \in \mathbb{Q}_p$ given by (1) let

$$s_1(\alpha) = s(\alpha) = \sum_{n=r}^{0} a_n p^n \quad \text{and} \quad s_1'(\alpha) = s_1(\alpha) - p \cdot \operatorname{sign}(s_1(\alpha)).$$

If $\alpha \in \mathbb{Q}_p \setminus p\mathbb{Z}_p$, then $s_1(\alpha)$ and $s_1'(\alpha)$ are representatives of α modulo $p\mathbb{Z}_p$ belonging to $\mathbb{Z}[\frac{1}{p}] \cap (-p, p)$. They have opposite signs. If $\alpha \in p\mathbb{Z}_p$, then $s_1(\alpha) = s_1'(\alpha) = 0$.

Similarly, let

$$s_2(\alpha) = \sum_{n=r}^{-1} a_n p^n \quad \text{and} \quad s_2'(\alpha) = s_2(\alpha) - \operatorname{sign}(s_2(\alpha)).$$

If $\alpha \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, then $s_2(\alpha)$ and $s_2'(\alpha)$ are representatives of α modulo $\mathbb{Z}_p$ belonging to $\mathbb{Z}[\frac{1}{p}] \cap (-1, 1)$. They have opposite signs. If $\alpha \in \mathbb{Z}_p$, then $s_2(\alpha) = s_2'(\alpha) = 0$.

In the algorithms below there are given some rules deciding which of these two representatives should be chosen in the definition of an appropriate continued fraction.

Algorithm II.

We use the above notation. Let $s_1'' = s_1$ and

$$s_2''(\alpha) = \begin{cases} s_2(\alpha) & \text{if} \quad v(\alpha - s_2(\alpha)) = 0, \\ s_2'(\alpha) & \text{otherwise.} \end{cases}$$

Then $v(\alpha - s_2''(\alpha)) = 0$, provided $v(\alpha) \leq 0$.

For a p-adic number α we define finite or infinite sequences (a_n) and (b_n) as follows.

Step 0. $a_0 = \alpha$, $b_0 = s_1''(a_0)$.

Step 1. If $a_0 = b_0$, then a_1 and b_1 are not defined.
If $a_0 \neq b_0$, then $a_1 = (a_0 - b_0)^{-1}$ and $b_1 = s_2''(a_1)$.

Step 2. If $a_1 = b_1$, then a_2 and b_2 are not defined.
If $a_1 \neq b_1$, then $a_2 = (a_1 - b_1)^{-1}$ and $b_2 = s_1''(a_2)$, etc.

We use s_1'' in steps of even number, and s_2'' in steps of odd number.

Obviously the sequence (b_n) satisfies (3), consequently by Lemma 1 the corresponding sequence $(\frac{A_n}{B_n})$ is convergent to a p-adic number. In fact $\lim \frac{A_n}{B_n} = \alpha$.

The proof is standard and we omit it (cf. [Br]).

Let us remark that, for n odd, we have $v(a_n - b_n) = v(a_n - s_2''(a_n)) = 0$. Consequently $v(a_{n+1}) = 0$, and hence

$$b_{n+1} = s_2''(a_{n+1}) \in \left\{\pm 1, \pm 2, \dots, \pm \frac{p-1}{2}\right\}.$$

We compare Algorithm II with the definition of Schneider. We use the notation

$$\alpha = b_0 + \cfrac{a_0}{b_1 + \cfrac{a_1}{b_2 + \cdots}} = \left[b_0; \begin{matrix} a_0 & a_1 & \cdots \\ b_1 & b_2 & \cdots \end{matrix}\right],$$

and $[b_0; b_1, b_2, \dots]_{\mathrm{II}}$ for the continued fraction obtained by Algorithm II.

Let us observe that, for an arbitrary $r \neq 0$,

$$\left[b_0; \begin{matrix} a_0 & \cdots & a_{k-1} & a_k & \cdots \\ b_1 & \cdots & b_k & b_{k+1} & \cdots \end{matrix}\right] = \left[b_0; \begin{matrix} a_0 & \cdots & ra_{k-1} & ra_k & \cdots \\ b_1 & \cdots & rb_k & b_{k+1} & \cdots \end{matrix}\right], \tag{5}$$

where the three elements indicated have been multiplied by r.

Let

$$\alpha = [b_0; b_1, b_2, \dots]_{\mathrm{II}} = \left[b_0; \begin{matrix} 1 & 1 & \cdots \\ b_1 & b_2 & \cdots \end{matrix}\right],$$

where $v(b_{2n}) = 0$ and $b_{2n-1} = \frac{c_{2n-1}}{p^{\alpha_n}}$, $|c_{2n-1}| < \frac{1}{2}p^{\alpha_n}$, for $n \geq 1$.

Then in view of (5)

$$\alpha = \left[b_0; \begin{matrix} p^{\alpha_1} & p^{\alpha_1} & p^{\alpha_2} & p^{\alpha_2} & \cdots \\ c_1 & b_2 & c_3 & b_4 & \cdots \end{matrix}\right].$$

In particular, if $\alpha_n = 1$ for all n, we get

$$[b_0; b_1, b_2, \dots]_{\mathrm{II}} = \left[b_0; \begin{matrix} p & p & p & p & \cdots \\ c_1 & b_2 & c_3 & b_4 & \cdots \end{matrix}\right],$$

and $b_j, c_k \in \left\{\pm 1, \pm 2, \dots, \pm \frac{p-1}{2}\right\}$.

Thus in this particular case we get a continued fraction of Schneider with the set $\{0, 1, \dots, p-1\}$ of residues modulo p replaced by $\left\{0, \pm 1, \pm 2, \dots, \pm \frac{p-1}{2}\right\}$.

Conversely, if the continued fraction of Schneider (with the set of residues changed as above)

$$\alpha = \left[b_0; \begin{matrix} p^{\alpha_1} & p^{\alpha_2} & \cdots \\ b_1 & b_2 & \cdots \end{matrix}\right]$$

satisfies $\alpha_{2n-1} = \alpha_{2n}$ for $n \geq 1$, then in view of (5) we get

$$\alpha = [b_0; b_1', b_2', \dots]_{\mathrm{II}},$$

where $b_{2n-1}' = b_{2n-1}/p^{\alpha_{2n-1}}$ and $b_{2n}' = b_{2n}$ for $n \geq 1$.

Algorithm III (For quadratic irrationalities only).

Let m be a rational number such that $\sqrt{m} \in \mathbb{Q}_p \setminus \mathbb{Q}$. Then every $\alpha \in \mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ can be written uniquely in the form $\alpha = \frac{\sqrt{m}+P}{Q}$ with $P, Q \in \mathbb{Q}$.

We use the above notation and proceed analogously as in Algorithm II, but we change s_1'' and s_2'' as follows. Let $s_1'''(\alpha) \in \{s_1(\alpha), s_1'(\alpha)\}$ and let $s_1'''(\alpha)$ have the same sign as PQ. Next,

$$s_2'''(\alpha) = \begin{cases} s_2(\alpha) & \text{if} \quad v(\alpha - s_2'(\alpha)) > 0, \\ s_2'(\alpha) & \text{if} \quad v(\alpha - s_2(\alpha)) > 0, \\ \tilde{s_2}(\alpha) & \text{otherwise,} \end{cases}$$

where $\tilde{s_2}(\alpha) \in \{s_2(\alpha), s_2'(\alpha)\}$ and $\tilde{s_2}(\alpha)$ has the same sign as PQ. We assume here that $PQ \neq 0$.

The algorithm is the same as Algorithm II but we replace s_1'' and s_2'' by s_1''' and s_2''', respectively.

Algorithm IV (For quadratic irrationalities only).

In Algorithm III we replace the conditions of the form $s \in \{t, t'\}$ and s has the same sign as PQ by the condition $s \in \{t, t'\}$ and $|\frac{P}{Q} - s| = \min(|\frac{P}{Q} - t|, |\frac{P}{Q} - t'|)$, where $|\cdot|$ means the ordinary absolute value in $\mathbb{R}$.

4. Numerical examples

To simplify notation, we use braces $\{\ \}$ to denote the period of continued fraction, and we use the symbol $\|$ if the period can be divided into some specific parts, e.g., symmetric ones. For example, $[a; b, c, d, c, e, c, d, c, e, c, d, c, e \dots]$ is denoted by $[a; b, \{c, d, c, \| e\}]$.

We use the notation $[b_0; b_1, \dots]_N$, where $N =$ I, II, III or IV, to indicate which algorithm has been used.

Usually we cannot prove that a continued fraction for a given quadratic irrationality is not periodic. We can state only that a period was not observed until, in the standard notation $a_n = \frac{\sqrt{m}+P_n}{Q_n}$, the integers P_n, Q_n have many digits (usually more than 10).

All computations have been performed using the package GP/PARI, version 1.39.

We have made the following observations concerning the periods. See also [Be1].

In most cases (especially if we use Algorithm I) the period consists of two parts of odd lengths, which are symmetric. In some cases the period consists of two parts of the same odd lengths, which differ slightly. There are also some periods without any regularity.

We cannot prove any of these observations. Of course if we use one of the Algorithms II–IV, the period length must be even by the construction provided the period exists.

Most examples with periods of length 2 are particular cases of the following lemma.

Lemma 2 (cf. [We2])). *Let $a, c, m \in \mathbb{Z}$, $p \nmid acm, b \in \mathbb{Z}[\frac{1}{p}]$, $v(b) < 0$, and m be not a square in $\mathbb{Z}$.*

Then

$$\sqrt{m} = [a; \{b, c\}]$$

if and only if $c = 2a$, $b = \frac{2a}{dp^k}$, for some $k \geq 1, d \in \mathbb{Z}, d|2a$, and

$$m = a^2 + dp^k.$$

Proof. The sufficiency of these conditions has been observed by de Weger [We2]. To prove the necessity we note that $x = \sqrt{m}$ satisfies

$$(x-a)(x-a+c) = \frac{c}{b}.$$

Since the left-hand side is an algebraic integer and $b = \frac{b'}{p^k}$ for some $b' \in \mathbb{Z}, p \nmid b'$, we get $b'|c$, $c = b'd$. Hence

$$x^2 + (c-2a)x - a(c-a) - p^k d = 0.$$

Since $\sqrt{m}$ is irrational, it follows that $c - 2a = 0$ and hence

$$m = x^2 = a(c-a) + p^k d = a^2 + dp^k.$$

Moreover $b = \frac{b'}{p^k} = \frac{2a}{dp^k}$. $\square$

Remark. In Lemma 2 we cannot omit the assumption that $\sqrt{m}$ is irrational, since, e.g.,

$$\frac{p+1}{2} = \left[\frac{p-1}{2}; \left\{\frac{p-1}{2p}, p-1\right\}\right].$$

It seems however that applying Algorithm II to a rational number α we always get a finite continued fraction, but we cannot prove this assertion.

Lemma 3. *If $a, b, c \in \mathbb{Z}$, $b' = \gcd(b, 2)$, and p is an odd prime number satisfying*

$$p \nmid abc, \ bb' \mid c + 2a, \ bc + 2p \mid c - 2a, \tag{6}$$

then

$$m := a^2 + p\frac{2a(bc+p) + pc}{b(bc+2p)} \in \mathbb{Z}. \tag{7}$$

Let $\alpha \in \mathbb{Q}_p$ be defined by the periodic continued fraction

$$\alpha = \left[a; \left\{\frac{b}{p}, c, \frac{b}{p}, \|2a\right\}\right]. \tag{8}$$

Then $\alpha^2 = m$, i.e., $\sqrt{m}$ in $\mathbb{Q}_p$ has a periodic continued fraction expansion given by (8). *The length of the period equals* 2 *if $c = 2a$ and $b \mid 2a$, and equals* 4 *otherwise.*

Proof. 1) From (6) and (7) it follows that $m \in \mathbb{Z}$ iff $b(bc+2p)$ divides $2a(bc+p)+pc$. We have by (6)

$$2a(bc+p) + pc = \begin{cases} 2a(bc+2p) + p(c-2a) & \equiv 0 \pmod{bc+2p}, \\ 2abc + p(c+2a) & \equiv 0 \pmod{bb'}. \end{cases} \tag{9}$$

Since $\gcd(b, bc+2p) = \gcd(b, 2) = b'$, from (9) it follows that $2a(bc+p) + pc \equiv 0 \pmod{b(bc+2p)}$.

2) The continued fraction in (8) is convergent in $\mathbb{Q}_p$ by Lemma 1. In a standard way one can find the quadratic equation satisfied by α. It is $\alpha^2 = m$, where m is given by (7).

The last part of the lemma is obvious. □

Taking some particular values for a, b, c satisfying (6) we get periodic continued fractions with the period of length 4, e.g.,

$$\sqrt{2p^2+2p+1} = \left[p+1; \left\{\frac{1}{p}, 1, \frac{1}{p}, \|2p+2\right\}\right],$$

$$\sqrt{(a+p)^2 - 2p^2} = \left[a; \left\{\frac{1}{p}, a-p, \frac{1}{p}, \|2a\right\}\right],$$

in particular taking the least $a > (\sqrt{2}-1)p$, we get, e.g., for $p = 257$

$$\sqrt{398} = \left[107; \left\{\frac{1}{p}, -150, \frac{1}{p}, \|214\right\}\right],$$

and for $p = 21961$

$$\sqrt{28322} = \left[9097; \left\{\frac{1}{p}, -12864, \frac{1}{p}, \|18194\right\}\right].$$

It is not clear if for given p there are infinitely many a, b, c satisfying (6).

We investigated continued fractions for square roots $\sqrt{m} \in \mathbb{Q}_p$, where $1 \le m \le 5000$, for $p = 5$, and $1 \le m \le 500$, for $p = 23, 257$, and 21961. We assumed always that $\gcd(m, p) = 1$. Then the Legendre symbol $(m/p) = 1$.

The results are as follows.

The field $\mathbb{Q}_5$.

Algorithm I.

The following continued fractions are periodic (we correct here the mistakes for $m = 11$ in [Br]). For brevity we give the results for $1 \le m \le 100$ only.

$$\sqrt{6} = \left[1; -\frac{8}{5}, \left\{\frac{6}{5}, \| \frac{7}{5}, -\frac{16}{25}, \frac{7}{5}\right\}\right]$$

$$\begin{aligned}\sqrt{11} = \Big[1; -\frac{9}{5}, \Big\{&\frac{9}{5}, -\frac{8}{5}, \frac{9}{5}, \| \frac{6}{5}, \frac{2}{5}, \frac{56}{25}, -\frac{2}{5}, \frac{6}{5}, \frac{27}{25}, -\frac{11}{5}, -\frac{2}{5}, \frac{16}{125},\\ &\frac{12}{5}, -\frac{58}{25}, \frac{12}{5}, \frac{16}{125}, -\frac{2}{5}, -\frac{11}{5}, \frac{27}{25}, \frac{6}{5}, -\frac{2}{5}, \frac{56}{25}, \frac{2}{5}, \frac{6}{5}\Big\}\Big]\end{aligned}$$

$$\sqrt{14} = \left[2; -\frac{3}{5}, \left\{-\frac{9}{5}, -\frac{6}{5}, \frac{166}{125}, -\frac{6}{5}, -\frac{9}{5}, \| -\frac{8}{5}\right\}\right]$$

$$\sqrt{21} = \left[1; \frac{3}{5}, \left\{\frac{3}{5}, -\frac{4}{5}, \frac{3}{5}, \| -\frac{7}{5}, \frac{26}{25}, -\frac{7}{5}\right\}\right]$$

$$\sqrt{24} = \left[2; -\frac{4}{5}, \left\{\frac{12}{5}, \| -\frac{9}{5}\right\}\right]$$

$$\sqrt{34} = \left[2; \frac{4}{5}, \left\{\frac{2}{5}, -\frac{28}{25}, \frac{2}{5}, \| -\frac{1}{5}, \frac{56}{25}, -\frac{1}{5}\right\}\right]$$

$$\sqrt{54} = \left[2; -\frac{23}{25}, \left\{-\frac{6}{5}, -\frac{1}{5}, \frac{9}{5}, \frac{4}{5}, -\frac{2}{5}, \frac{4}{5}, \frac{9}{5}, -\frac{1}{5}, -\frac{6}{5}, \| -\frac{48}{25}\right\}\right]$$

$$\sqrt{69} = \left[2; \frac{3}{5}, \left\{-\frac{6}{5}, -\frac{1}{5}, -\frac{8}{5}, -\frac{1}{5}, -\frac{6}{5}, \| -\frac{2}{5}, -\frac{2}{5}, \frac{174}{125}, -\frac{2}{5}, -\frac{2}{5}\right\}\right]$$

$$\begin{aligned}\sqrt{74} = \Big[2; \frac{6}{5}, \Big\{&-\frac{6}{5}, \frac{7}{5}, \frac{36}{25}, \frac{7}{5}, -\frac{6}{5}, \| \frac{1}{5}, -\frac{3}{5}, -\frac{23}{25}, -\frac{2}{5}, \frac{4}{5}, -\frac{7}{5}, \frac{8}{5},\\ &-\frac{8}{25}, -\frac{3}{5}, \frac{9}{5}, \frac{52}{25}, -\frac{21}{25}, \frac{4}{5}, \frac{1}{5}, \frac{11}{5}, -\frac{3}{5}, \frac{156}{125}, -\frac{11}{5},\\ &\frac{1}{5}, \frac{8}{5}, -\frac{11}{25}, \frac{4}{5}, -\frac{7}{5}, -\frac{3}{5}, -\frac{36}{25}, -\frac{4}{5}, -\frac{9}{5}, \frac{14}{25}, -\frac{9}{5},\\ &-\frac{4}{5}, -\frac{36}{25}, -\frac{3}{5}, -\frac{7}{5}, \frac{4}{5}, -\frac{11}{25}, \frac{8}{5}, \frac{1}{5}, -\frac{11}{5}, \frac{156}{125}, -\frac{3}{5},\\ &\frac{11}{5}, \frac{1}{5}, \frac{4}{5}, -\frac{21}{25}, \frac{52}{25}, \frac{9}{5}, -\frac{3}{5}, -\frac{8}{25}, \frac{8}{5}, -\frac{7}{5}, \frac{4}{5}, -\frac{2}{5}, -\frac{23}{25}, -\frac{3}{5}, \frac{1}{5}\Big\}\Big]\end{aligned}$$

$$\begin{aligned}\sqrt{76} = \Big[1; \frac{34}{25}, \Big\{&-\frac{1}{5}, -\frac{6}{5}, -\frac{9}{5}, -\frac{2}{5}, -\frac{9}{5}, -\frac{6}{5}, -\frac{1}{5}, \| -\frac{16}{25}, -\frac{3}{5}, -\frac{9}{5}, -\frac{1}{5},\\ &\frac{12}{5}, -\frac{12}{5}, \frac{2}{5}, \frac{8}{5}, \frac{9}{5}, -\frac{2}{5}, \frac{9}{5}, \frac{8}{5}, \frac{2}{5}, -\frac{12}{5}, \frac{12}{5}, -\frac{1}{5}, -\frac{9}{5}, -\frac{3}{5}, -\frac{16}{25}\Big\}\Big]\end{aligned}$$

$$\sqrt{94} = \left[2; -\frac{2}{5}, \left\{-\frac{8}{5}, \frac{62}{125}, -\frac{8}{5}, \| -\frac{7}{5}, \frac{3}{5}, \frac{8}{5}, \frac{1}{5}, \frac{12}{5}, \frac{1}{5}, \frac{8}{5}, \frac{3}{5}, -\frac{7}{5}\right\}\right]$$

$$\sqrt{99} = \left[2; \frac{11}{5}, \left\{-\frac{23}{25}, \| \frac{6}{5}, \frac{2}{5}, \frac{2}{5}, \frac{43}{25}, \frac{2}{5}, \frac{2}{5}, \frac{6}{5}\right\}\right]$$

For other m's in question, i.e., for $m = 19, 26, 29, 31, 39, 41, 44, 46, 51, 56, 59, 61, 66, 71, 79, 84, 86, 89, 91, 96$ (20 cases) no period has been observed.

Algorithm II gives similar results as Algorithm I. In many cases periods are shorter. We give the results for $1 \le m \le 100$ only.

$$\sqrt{6} = \left[1; \left\{\frac{2}{5}, 2\right\}\right]$$

$$\sqrt{11} = \left[1; \left\{\frac{1}{5}, 2\right\}\right]$$

$$\sqrt{14} = \left[2; \left\{\frac{2}{5}, \| -1, -\frac{1}{5}, 2, -\frac{1}{5}, -1\right\}\right]$$

$$\sqrt{21} = \left[1; \left\{-\frac{2}{5}, 1, \frac{2}{5}, 2, \frac{1}{5}, -2, \frac{2}{5}, -2, \frac{1}{5}, 2, \frac{2}{5}, 1, -\frac{2}{5}, \|2\right\}\right]$$

$$\sqrt{24} = \left[2; \left\{\frac{1}{5}, \| -1, -\frac{2}{5}, -1\right\}\right]$$

$$\sqrt{26} = \left[1; \left\{\frac{2}{25}, 2\right\}\right]$$

$$\sqrt{29} = \left[2; \left\{\frac{4}{25}, \| -1, \frac{1}{5}, 1, \frac{2}{5}, 2, -\frac{3}{25}, -1, -\frac{46}{125}, -1, \frac{-3}{25}, 2, \frac{2}{5}, 1, \frac{1}{5}, -1\right\}\right]$$

$$\sqrt{31} = \left[1; \left\{\frac{2}{5}, 1, \frac{2}{5}, 1, -\frac{3}{25}, -2, \frac{2}{5}, -2, \frac{2}{5}, -2, -\frac{3}{25}, 1, \frac{2}{5}, 1, \frac{2}{5}, \|2\right\}\right]$$

$$\begin{aligned}\sqrt{34} = \Bigg[2; \Bigg\{&-\frac{1}{5}, 1, -\frac{2}{5}, -1, -\frac{28}{25}, 1, -\frac{2}{5}, -1, -\frac{1}{5}, 1, -\frac{6}{25}, -2, -\frac{3}{5}, 1, -\frac{2}{5}, -1, \frac{1}{5}, 2, -\frac{9}{25}, 1, \frac{2}{5},\\ &-2, \frac{2}{5}, -2, \frac{29}{125}, -2, -\frac{1}{5}, -1, \frac{12}{25}, -2, -\frac{7}{25}, -2, -\frac{9}{5}, -2, -\frac{4}{5}, 1, -\frac{2}{5}, -1, \frac{2}{5}, 2, \frac{19}{25}, -1\Bigg\}\Bigg]\end{aligned}$$

$$\sqrt{39} = \left[2; \left\{\frac{2}{5}, 2, \frac{1}{5}, -1, \frac{2}{5}, 2, \frac{2}{5}, -2, \frac{28}{25}, -1, -\frac{1}{5}, -1, \frac{6}{25}, 2, -\frac{1}{5}, 2, \frac{4}{5}, -1\right\}\right]$$

$$\sqrt{51} = \left[1; \left\{\frac{1}{25}, 2\right\}\right]$$

$$\sqrt{54} = \left[2; \left\{\frac{2}{25}, \| -1, \frac{1}{5}, -2, -\frac{1}{5}, -1, -\frac{2}{5}, 1, -\frac{6}{5}, 1, -\frac{2}{5}, -1, -\frac{1}{5}, -2, \frac{1}{5}, -1\right\}\right]$$

$$\sqrt{56} = \left[1; \left\{\frac{2}{5}, -1, -\frac{2}{5}, 2, -\frac{1}{5}, 2, -\frac{2}{5}, -1, \frac{2}{5}, \|2\right\}\right]$$

$$\sqrt{61} = \left[1; \left\{\frac{1}{5}, -2, -\frac{2}{5}, -1, -\frac{1}{5}, -2, -\frac{2}{5}, 1, \frac{2}{5}, 1, -\frac{2}{5}, -2, -\frac{1}{5}, -1, -\frac{2}{5}, -2, \frac{1}{5}, \|2\right\}\right]$$

$$\sqrt{69} = \left[2; -\frac{2}{25}, 1, \frac{6}{5}, \left\{-1, -\frac{1}{5}, 1, -\frac{2}{5}, 1, -\frac{1}{5}, -1, \|\frac{1}{5}, -2, \frac{1}{5}\right\}\right]$$

$$\sqrt{79} = \left[2; -\frac{7}{25}, 1, \frac{2}{5}, 2, -\frac{2}{5}, 1, \frac{1}{5}, 2, -\frac{4}{25}, 2, \frac{1}{5}, 1, -\frac{2}{5}, 2, \frac{2}{5}, \left\{1, -\frac{7}{25}, -1, \frac{1}{5}, 2, \frac{9}{25}, -1 - \frac{3}{5}\right\}\right]$$

$$\sqrt{84} = \left[2; \left\{-\frac{1}{5}, 2, \frac{1}{5}, -1, \frac{2}{5}, -1, \frac{1}{5}, 2, -\frac{1}{5}, \| -1, -\frac{2}{25}, -1\right\}\right]$$

$$\sqrt{99} = \left[2; \left\{\frac{1}{5}, -2, \frac{2}{5}, 21, -\frac{2}{5}, -2, -\frac{2}{5}, -1, \frac{1}{5}, 1, -\frac{1}{5}, 2, \frac{32}{25}, -1, \frac{2}{5}, 1, -\frac{2}{5}, -1\right\}\right]$$

In most cases the period begins at b_1, but there are exceptions: $m = 69, 79$, when the period begins at b_4, resp. b_{16}. In most cases the period has some symmetry (the exceptions are $m = 34, 39, 79, 99$).

No period has been observed for $m =$ 19, 41, 44, 46, 59, 66, 71, 74, 76, 86, 89, 91, 94, 96 (14 cases).

Algorithms III and **IV** give very similar results, so we present them simultaneously (for $1 \le m \le 50$ only).

$$\sqrt{6} = \left[1; \left\{\frac{2}{5}, 2\right\}\right]_{\text{II,III,IV}}$$
$$\sqrt{11} = \left[1; \left\{\frac{1}{5}, 2\right\}\right]_{\text{II,III,IV}}$$
$$\sqrt{14} = \left[2; \left\{\frac{2}{5}, 4\right\}\right]_{\text{III,IV}}$$
$$\sqrt{19} = \left[2; \left\{\frac{3}{5}, -2, \frac{1}{5}, \| -3, \frac{2}{5}, -1\right\}\right]_{\text{III,IV}}$$
$$\sqrt{21} = \left[1; \left\{-\frac{2}{5}, 1, \frac{2}{5}, -3\right\}\right]_{\text{III,IV}}$$
$$\sqrt{24} = \left[2; \left\{\frac{1}{5}, 4\right\}\right]_{\text{III,IV}}$$
$$\sqrt{26} = \left[1; \left\{\frac{2}{25}, 2\right\}\right]_{\text{III,IV}}$$
$$\sqrt{29} = \left[2; \left\{\frac{4}{25}, 4\right\}\right]_{\text{III,IV}}$$

The first case where the Algorithms III and IV give distinct results is $m = 31$:

$$\sqrt{31} = \left[1; \left\{\frac{2}{5}, -4, -\frac{1}{5}, -4, \frac{2}{5}, \|2\right\}\right]_{\text{III}}$$
$$\sqrt{31} = \left[1; \left\{\frac{2}{5}, -4, -\frac{1}{5}, 1, \frac{3}{5}, -2, \frac{4}{5}, 1, -\frac{1}{5}, -3\right\}\right]_{\text{IV}}$$
$$\sqrt{34} = \left[2; \left\{-\frac{1}{5}, 1, \frac{3}{5}, 2, \frac{3}{5}, -4, \frac{6}{25}, 1, \|\frac{1}{5}, -1, -\frac{3}{5}, -2, -\frac{3}{5}, 4, -\frac{6}{25}, -1\right\}\right]_{\text{III,IV}}$$
$$\sqrt{39} = \left[2; \left\{\frac{2}{5}, -3, \frac{3}{5}, 2\right\}\right]_{\text{III,IV}}$$

For $m = 41, 44, 46$ the results of Algorithms III and IV differ.

$$\begin{aligned}\sqrt{41} = \Bigg[1; \Bigg\{&\frac{4}{5}, -4, -\frac{4}{5}, 1, -\frac{3}{5}, -3, \frac{13}{25}, -2, \frac{2}{5}, -1, -\frac{3}{5}, -1, \frac{2}{5}, 4,\\ &-\frac{3}{25}, -1, -\frac{1}{5}, 3, -\frac{2}{5}, 1, \frac{3}{5}, 1, \frac{2}{5}, -4, -\frac{3}{5}, 2, -\frac{8}{25}, -3\Bigg\}\Bigg]_{\text{III}}\end{aligned}$$
$$\sqrt{41} = \left[1; \left\{-\frac{1}{5}, 3, \frac{3}{5}, -1, \frac{2}{5}, -1, \frac{3}{5}, -2, \frac{8}{25}, 3, \|\frac{1}{5}, -3, -\frac{3}{5}, 1, -\frac{2}{5}, 1, -\frac{3}{5}, 2, -\frac{8}{25}, -3\right\}\right]_{\text{IV}}$$
$$\sqrt{44} = \left[2; \left\{\frac{3}{5}, -4, \frac{3}{5}, \|4\right\}\right]_{\text{III}}$$
$$\sqrt{44} = \left[2; \left\{-\frac{2}{5}, 3, \frac{1}{5}, \|2, -\frac{3}{5}, -1\right\}\right]_{\text{IV}}$$
$$\sqrt{46} = \left[1; \frac{3}{5}, -3, \frac{4}{5}, \left\{-4, \| -\frac{1}{5}, 4, -\frac{22}{25}, 4, -\frac{1}{5}\right\}\right]_{\text{III}}$$
$$\sqrt{46} = \left[1; -\frac{2}{5}, 4, -\frac{3}{5}, 1, \left\{-\frac{1}{5}, -1, -\frac{4}{5}, -1, -\frac{1}{5}, \| -4\right\}\right]_{\text{IV}}$$

We have verified that Algorithm IV applied to $\sqrt{m}$ gives periodic continued fractions for every m, $2 \le m \le 5000$. For example, for $m = 3994$ the period begins with b_{61} and has 160 terms.

The field $\mathbb{Q}_{23}$.

We have applied Algorithm I to $\sqrt{m}$, for $1 \le m \le 200$ and the period has been observed in three cases only.

$$\sqrt{75} = \left[11; -\frac{34}{23}, \left\{\frac{38}{23}, -\frac{57}{23}\right\}\right]_{\mathrm{I}}$$
$$\sqrt{98} = \left[11; -\frac{45}{23}, \left\{\frac{34}{23}, -\frac{68}{23}\right\}\right]_{\mathrm{I}}$$
$$\sqrt{167} = \left[11; -\frac{12}{23}, \left\{\frac{70}{23}, -\frac{35}{23}\right\}\right]_{\mathrm{I}}$$

On the other hand applying Algorithm IV for $2 \le m \le 500$ we get periods in most cases. The only exceptions for $m \le 200$ where the period has not been observed are $m = 93, 101, 117, 119, 133, 141, 154, 163, 186$. We have the following examples.

$$\sqrt{2} = \left[5; \left\{-\frac{10}{23}, 10\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{3} = \left[7; \left\{-\frac{7}{23}, 14\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{6} = \left[11; -\frac{9}{23}, -7, \frac{1}{23}, 15, \left\{\frac{12}{23}, -8\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{29} = \left[11; \left\{\frac{6}{23}, 7, -\frac{13}{23}, 13, -\frac{10}{23}, 3, -\frac{22}{23}, 10, -\frac{113}{529}, \dots, 7, \frac{6}{23}, 22\right\}\right]_{\mathrm{IV}}$$

The period has length 94 and no symmetry has been observed.

The longest period of 112 terms has $\sqrt{462}$.

$$\sqrt{462} = \left[5; \frac{9}{23}, \left\{8, \frac{7}{23}, \dots, -\frac{14}{23}\right\}\right]_{\mathrm{IV}}$$

The field $\mathbb{Q}_{257}$.

We have applied Algorithm IV. For $1 \le m \le 500$, the continued fractions were periodic in 59 cases. The length of the period was 2 (for $m = 15, 32, 62, 67, \ldots$), 4 or 10. In the case of length 2 (resp. 4) the continued fraction has the form described in Lemma 2 (resp. in Lemma 3).

We write down all the examples with period lenghts 4 and 10.

$$\sqrt{50} = \left[43; \left\{-\frac{49}{257}, 12, -\frac{49}{257}, \|86\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{120} = \left[67; \left\{-\frac{23}{257}, 27, -\frac{23}{257}, \|134\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{241} = \left[64; \left\{\frac{60}{257}, -8, \frac{60}{257}, \|128\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{392} = \left[69; \left\{\frac{7}{257}, -47, \frac{7}{257}, \|138\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{396} = \left[41; \left\{\frac{35}{257}, -12, \frac{35}{257}, \|82\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{398} = \left[107; \left\{\frac{1}{257}, -150, \frac{1}{257}, \|214\right\}\right]_{\mathrm{IV}}$$
$$\sqrt{454} = \left[35; \left\{-\frac{109}{257}, 24, \frac{26}{257}, 1, -\frac{17032}{66049}, 1, \frac{26}{257}, 24, -\frac{109}{257}, \|70\right\}\right]_{\mathrm{IV}}$$

The field $\mathbb{Q}_{21961}$.

We have applied Algorithm IV to $\sqrt{m}$ for $1 \le m \le 500$, and continued fractions were periodic only for $m = 3, 57, 178, 228, 240$, and 363. The continued fraction always has the form described in Lemma 2:

$$\sqrt{a^2+dp}=[a;\{2a/dp,2a\}],$$

where $p=21961$ and the integers a,d satisfy $d\,|\,2a$, and $m=a^2+dp$. Namely,

m	a	$2a/d$
3	363	−121
57	646	−68
178	2120	−210
228	1292	−34
240	149	−298
363	3993	−11

In other cases the period has not been observed.

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