Math 280 Solutions for September 5

Pythagoras Level

Problem 1: [Nick's Math Puzzles #5] **Solution by Diophantine Equation**

Let x be the number of dollars in the check, and y be the number of cents. Then $100y + x - 50 = 3(100x + y)$. Therefore $97y - 299x = 50$.

A standard solution to this type of linear Diophantine equation uses Euclid's algorithm.

The Euclidean algorithm shows that gcd(97,299) = 1.

To solve $97y - 299x = gcd(97, 299) = 1$, we can proceed backwards through the Euclidean algorithm and arrive at: $1 = 37 * 97 - 12 * 299$

Therefore a solution to $97y - 299x = 1$ is $y = 37$, $x = 12$. Hence a solution to $97y - 299x = 50$ is $y = 50 * 37 = 1850$, $x = 50 * 12 = 600$. It can be shown that all integer solutions of $97y - 299x = 50$ are of the form $y = 1850 + 299k$, $x = 600 + 97k$, where k is any integer.

In this case, because x and y must be between 0 and 99, we choose $k = -6$. This gives $y = 56$, $x = 18$. So the check was for \$18.56.

Solution by Simultaneous Equations

Let x be the number of dollars in the check, and y be the number of cents. Consider the numbers of dollars and cents Ms Smith holds at various times. The original check is for x dollars and y cents. The bank teller gave her y dollars and x cents. After buying the newspaper she has y dollars and $x - 50$ cents. We are also told that after buying the newspaper she has three times the amount of the original check; that is, $3x$ dollars and $3y$ cents.

Clearly (y dollars plus $x - 50$ cents) equals ($3x$ dollars plus $3y$ cents). Then, bearing in mind that x and y must both be less than 100 (for the teller's error to make sense), we equate dollars and cents.

As $-50 \leq (x - 50) \leq 49$ and $0 \leq 3y \leq 297$, there is a relatively small number of ways in which we can equate dollars and cents. (If there were many different ways, this whole approach would not be viable.) Clearly, $3y - (x - 50)$ must be divisible by 100. Further, by the above inequalities, $-49 \leq 3y - (x - 50) \leq 347$, giving us four multiples of 100 to check.

If $3y - (x - 50) = 0$, then we must have $3x = y$, giving $x = -25/4$, $y = -75/4$.
If $3y - (x - 50) = 100$, then (to balance) we must have $3x - y = -1$, giving $x = 47/8$, $y = 149/8$.
If $3y - (x - 50) = 200$, then we must have $3x - y = -2$, giving $x = 18$, $y = 56$.
If $3y - (x - 50) = 300$, then we must have $3x - y = -3$, giving $x = 241/8$, $y = 747/8$.

There is only one integer solution; so the check was for \$18.56.

Problem 2: [Nick's Math Puzzle #7] Let the original pile have n coconuts. Let a be the number of coconuts in each of the five piles made by the first man, b the number of coconuts in each of the five piles made by the second man, and so on.

Writing a Diophantine equation to represent the actions of each man, we have
$n = 5a + 1$ if and only if $n + 4 = 5(a + 1)$
$4a = 5b + 1$ if and only if $4(a + 1) = 5(b + 1)$
$4b = 5c + 1$ if and only if $4(b + 1) = 5(c + 1)$
$4c = 5d + 1$ if and only if $4(c + 1) = 5(d + 1)$
$4d = 5e + 1$ if and only if $4(d + 1) = 5(e + 1)$

Hence $n + 4 = 5 * (5/4)^4(e + 1)$, and so $n = (5^5/4^4)(e + 1) - 4$.

Note that, since 5 and 4 are relatively prime, $5^5/4^4 = 3125/256$ is a fraction in its lowest terms. Hence the only integer solutions of the above equation are where $e + 1$ is a multiple of 44, whereupon $d + 1$, $c + 1$, $b + 1$, and $a + 1$ are all integers.

So the general solution is $n = 3125r - 4$, where r is a positive integer, giving a smallest solution of 3121 coconuts in the original pile.

Newton Level

Problem 3: [MAA-NCS 2007 #4] The value is $\sqrt{2} + \sqrt{3} - 7/3$. Note that $< x + n > = < x >$ for every real x and every integer n, so $< x^2 + 2x - 3 > = < (x + 1)^2 >$. Using the substitution $u = x + 1$ then we have

$$\int_{-1}^{1} < x^2 + 2x - 3 > dx = \int_{-1}^{1} < (x + 1)^2 > dx = \int_{0}^{2} < u^2 > du.$$

For $0 \leq u < 1$, we have $0 \leq u^2 < 1$ and $< u^2 > = u^2$. For $1 \leq u < \sqrt{2}$, we have $1 \leq u^2 < 2$ and $< u^2 > = u^2 - 1$. For

$\sqrt{2} \le u < \sqrt{3}$, we have $< u^2 >= u^2 - 2$, and for $\sqrt{3} \le u < 2$, we have $< u^2 >= u^2 - 3$. Thus

$$\begin{aligned}
\int_0^2 < u^2 > du &= \int_0^1 u^2 du + \int_1^{\sqrt{2}} (u^2 - 1) du + \int_{\sqrt{2}}^{\sqrt{3}} (u^2 - 2) du + \int_{\sqrt{3}}^2 (u^2 - 3) du \\
&= \int_0^2 u^2 du - 1(\sqrt{2} - 1) - 2(\sqrt{3} - \sqrt{2}) - 3(2 - \sqrt{3}) \\
&= \frac{8}{3} + 1 - 6 + \sqrt{2} + \sqrt{3} \\
&= \sqrt{2} + \sqrt{3} - 7/3
\end{aligned}$$

Problem 4: [MAA-NCS 2005 #8] Here one should try to find a pattern in the partial sums, and prove it by induction. The first few partial sums are

$$S_1 = \frac{1}{3}, \quad S_2 = \frac{7}{3 \cdot 5}, \quad S_3 = \frac{52}{3 \cdot 4 \cdot 7}$$

The pattern can be seen most readily by comparing with the asserted limit, $1/2$. Look at the differences $1/2 - S_n$. Thus we find

$$S_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right), \quad S_2 = \frac{1}{2}\left(1 - \frac{1}{3 \cdot 5}\right), \quad S_3 = \frac{1}{2}\left(1 - \frac{1}{3 \cdot 5 \cdot 7}\right)$$

suggesting the formula

$$S_n = \frac{1}{2}\left(1 - \frac{1}{3 \cdot 5 \cdots (2n+1)}\right).$$

This is easily proved by induction. We have it for $n = 1$. Suppose it holds for a particular n. Then

$$\begin{aligned}
S_{n+1} &= S_n + \frac{n+1}{3 \cdot 5 \cdots (2n+1)(2n+3)} \\
&= \frac{1}{2}\left(1 - \frac{1}{3 \cdot 5 \cdots (2n+1)}\right) + \frac{n+1}{3 \cdot 5 \cdots (2n+1)(2n+3)} \\
&= \frac{1}{2}\left(1 - \frac{2n+3}{3 \cdot 5 \cdots (2n+1)(2n+3)} + \frac{2n+2}{3 \cdot 5 \cdots (2n+1)(2n+3)}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{3 \cdot 5 \cdots (2n+3)}\right)
\end{aligned}$$

By induction the formula for S_n holds and now the limit of the partial sums is seen to be $1/2$.

Wiles Level

Problem 5: [Putnam 2006 A1]

$$(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2).$$

We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2 + y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \le 6r,$$

or

$$(r - 3)^2 + z^2 \le 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2 + z^2 \le 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

Problem 6: [Putnam 2006 B5] The answer is $1/16$. We have

$$\begin{aligned}
&\int_0^1 x^2 f(x)\, dx - \int_0^1 x f(x)^2\, dx \\
&= \int_0^1 (x^3/4 - x(f(x) - x/2)^2)\, dx \\
&\le \int_0^1 x^3/4\, dx = 1/16,
\end{aligned}$$

with equality when $f(x) = x/2$.