Math 280 Problems for September 14

**Problem 1:** Given distinct points $a_1 < a_2 < a_3 < \cdots < a_{100}$ on the real line, determine, with proof, the exact set of real numbers $x$ for which the sum

$$
\sum_{i=1}^{100} |x - a_i|
$$

takes its minimal value.

**Problem 2:** Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let a new sequence $q_1, q_2, q_3, \ldots$ be defined by $q_1 = a_1$, $q_2 = a_2q_1 + 1$, and $q_n = a_nq_{n+1} + q_{n+2}$ for $n \geq 3$. Prove that no two consecutive $q_n$’s are even.

**Problem 3:** A function $f(n)$ is defined for all positive integers $n$ as follows: First add the digits of $n$ (in decimal notation) to get a number $n_1$, say; then add the digits of $n_1$ to get $n_2$; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$. Thus, for example, $f(989) = 8$, since $9 + 8 + 9 = 26, 2 + 6 = 8$. Prove that, for all positive integers $n$, $f(123456789n) = f(n)$.

**Problem 4:** Given a nonnegative integer $n$, let $\hat{n}$ denote the integer obtained by reversing the digits of $n$ in the standard decimal representation; for example, $\hat{935} = 539$. Let $f(n) = n + \hat{n}$, $g(n) = n - \hat{n}$, and $h(n) = f(g(n))$. For example, if $n = 935$, then $g(n) = 935 - 539 = 396$, and $h(n) = f(396) = 396 + 693 = 1089$ Prove that $h(n) = 1089$ for all three digit integers $n$ whose first digit exceeds the last digit by at least 2.

**Problem 5:** A polynomial $P(x)$ is known to be of the form

$$
P(x) = x^{15} - 9x^{14} + \cdots - 7,
$$

where the ellipsis ($\cdots$) represents unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots of $P(x)$.

**Problem 6:** Does there exist a multiple of 2008 whose decimal representation involves only a single digit (such as 11111 or 22222222)?