Math 280 Problems for September 14

Problem 1: Given distinct points $a_1 < a_2 < a_3 < \cdots < a_{100}$ on the real line, determine, with proof, the exact set of real numbers $x$ for which the sum

$$
\sum_{i=1}^{100} |x - a_i|
$$

takes its minimal value.

[UIUC 2011 #1] Let $S(x) = \sum_{i=1}^{100} |x - a_i|$ the sum we seek to minimize. By the triangle inequality, we have, for any $i \in \{1, 2, \ldots, 100\}$,

$$
|a_i - a_{101-i}| = |a_i - x + x - a_{101-i}| \leq |a_i - x| + |x - a_{101-i}|
$$

Summing over $i = 1, \ldots, 50$ we get

$$
\sum_{i=1}^{50} |a_i - a_{101-i}| \leq S(x)
$$

The real numbers $x$ that minimize $S(x)$ are exactly those $x$ for which equality holds. However equality holds in the triangle inequality if and only if $a_i - x$ and $x - a_{101-i}$ have the same sign, i.e., if and only if $x$ lies between $a_i$ and $a_{101-i}$. Hence equality for the sum holds if and only if $x$ lies between $a_i$ and $a_{101-i}$ for each $i = 1, 2, \ldots, 50$ i.e., if and only if $a_{50} \leq x \leq a_{51}$. Hence the real numbers that minimize $S(x)$ are exactly those in the interval $a_{50} \leq x \leq a_{51}$.

Problem 2: Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let a new sequence $q_1, q_2, q_3, \ldots$ be defined by $q_1 = a_1, q_2 = a_2 q_1 + 1$, and $q_n = a_n q_{n-1} + q_{n+1}$ for $n \geq 3$. Prove that no two consecutive $q_n$’s are even.

[UIUC 2010 #1] We argue by contradiction. Suppose there exist pairs of consecutive $q_n$ ’s that are both even. Among these let $(q_i, q_{i+1})$ be the pair with smallest index $i$. First note that, if $q_i$ is even, then $q_2 = a_2 q_1 + 1$ is odd. Thus $q_1$ and $q_2$ cannot both be even, so the minimal index $i$ such that $q_i$ and $q_{i+1}$ are both even must be at least 2. By the given recurrence we have $q_{i+1} = q_{i+1} - a_i q_i + q_i$, so $q_{i+1}$ is the difference of two even numbers and therefore must itself be even. Hence $(q_{i+1}, q_i)$ is a pair of consecutive even terms among the $q_n$ ’s, contradicting the minimality of $i$. Thus there do not exist consecutive even members of the sequence.

Problem 3: A function $f(n)$ is defined for all positive integers $n$ as follows: First add the digits of $n$ (in decimal notation) to get a number $n_1$, say; then add the digits of $n_1$ to get $n_2$; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$. Thus, for example, $f(989) = 8$, since $9 + 8 + 9 = 26, 2 + 6 = 8$. Prove that, for all positive integers $n$, $f(1234567n) = f(n)$.

[UIUC 2010 #2] We use congruences modulo 9. By an extension of the test for divisibility by 9, any positive integer is congruent modulo 9 to the sum of its decimal digits. Since $f(n)$ is obtained by an iteration of the “sum of digits” function, it follows that $f(n)$ satisfies the congruence $f(n) \equiv n \mod 9$. Moreover, since $f(n)$ is in the set $\{1, 2, \ldots, 9\}$, $f(n)$ is uniquely defined by its congruence modulo 9. Thus, to prove the claim, it suffices to show that, for all positive integers $n$,

$$
1234567 \cdot n \equiv n \mod 9
$$

But the latter follows from the fact that the number 1234567 is congruent to $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 \equiv 1 \mod 9$.

Problem 4: Given a nonnegative integer $n$, let $\tilde{n}$ denote the integer obtained by reversing the digits of $n$ in the standard decimal representation: for example, 935 is 539. Let $f(n) = n + \tilde{n}$, $g(n) = n - \tilde{n}$, and $h(n) = f(g(n))$. For example, if $n = 935$, then $g(n) = 935 - 539 = 396$, and $h(n) = f(396) = 396 + 693 = 1089$. Prove that $h(n) = 1089$ for all three digit integers $n$ whose first digit exceeds the last digit by at least 2.

[UIUC 2009 #1] Let $n$ denote an integer of the given form, i.e., $n = a_2 a_1 a_0$ with $a_2 \geq a_0 + 2$. Then

$$
n = 100 a_2 + 10 a_1 + a_0, \quad \tilde{n} = 100 a_0 + 10 a_1 + a_2, \quad g(n) = n - \tilde{n} = 100 \cdot (a_2 - a_0) - (a_2 - a_0) = 100 \cdot (a_2 - a_0 - 1) + 10 \cdot 9 + 1 \cdot (10 - a_2 + a_0), \quad \tilde{g}(n) = 100 \cdot (10 - a_2 + a_0) + 10 \cdot 9 + 1 \cdot (a_2 - a_0 - 1), \quad h(n) = g(n) + \tilde{g}(n) = 100 \cdot (10 - 1) + 10 \cdot (9 + 9) + (10 - 1) = 1089,
as claimed. Note that the given condition on the first and last digits of \( n \), namely \( a_2 - a_0 \geq 2 \), ensures that the coefficients \( a_2 - a_0 - 1 \) and \( 10 - a_2 + a_0 \) in the above expressions are integers in the interval \([1, 9]\), so these expressions indeed represent proper decimal expansions.

**Problem 5:** A polynomial \( P(x) \) is known to be of the form

\[
P(x) = x^{15} - 9x^{14} + \cdots + 7
\]

where the ellipsis (\( \cdots \)) represents unknown intermediate terms. It is also known that all roots of \( P(x) \) are integers. Find the roots of \( P(x) \).

[UIUC 2009 #4] Since \( P(x) \) has degree 15, it has 15 roots (counted with multiplicity). Let \( r_1, r_2, ..., r_{15} \) denote these roots, which, by assumption, are all integers. Since \( P(x) \) has leading term 1, it can be written as

\[
P(x) = \prod_{i=1}^{15} (x - r_i).
\]

Expanding this product we obtain

\[
P(x) = x^{15} + \left( \sum_{i=1}^{15} (-r_i) \right) x^{14} + \cdots + \prod_{i=1}^{15} (-r_i).
\]

Comparing this expression with the given form of \( P(x) \), we get

\[
\sum_{i=1}^{15} (r_i) = 9
\]

and

\[
\prod_{i=1}^{15} (r_i) = 7.
\]

The product forces one of the roots to be 7 or \(-7\), and the remaining 14 roots to be 1 or \(-1\). However, in the case when one of the roots is \(-7\) the sum of all roots can be at most \(-7 + 14 = 7\), contradicting the sum requirement. Hence one root must be 7, and the other 14 roots must be 1 or \(-1\). Inspection finds 1 of multiplicity 8 and \(-1\) of multiplicity 6 gives the desired sum. So the roots are

\[
7, 1, \ldots, 1, -1, \ldots, -1
\]

\[8 \text{ times} \quad 6 \text{ times}\]

**Problem 6:** Does there exist a multiple of 2008 whose decimal representation involves only a single digit (such as 11111 or 22222222)?

[UIUC 2008 #1] The answer is yes; specifically, we will show that there exists a multiple of 2008 of the form 888 \ldots 8. Given a digit \( d \in \{1, 2, \ldots, 9\} \), let \( N_{d,k} \) be the number whose decimal representation consists of \( k \) digits \( d \). Note that

\[
N_{d,k} = d \sum_{i=0}^{k-1} 10^i = \frac{d(10^k - 1)}{9}
\]

Thus, a given positive integer \( m \) has a multiple of this form if and only if the congruence (\(^*\)) \( d(10^k - 1) \equiv 0 \mod 9m \) has a solution \( k \). We apply this with \( d = 8 \) and \( m = 2008 \). Then (\(^*\)) is equivalent to (\(^*\)) \( 10^k - 1 \equiv 0 \mod 9 \cdot 2008/8 = 9251 \). Since \( 10^k \equiv 1^k \equiv 1 \mod 9 \) for any positive integer \( k \), (\(^*\)) is equivalent to (\(^*\)) \( 10^k \equiv 1 \mod 251 \). Now, 251 is prime, so by Fermats Theorem, we have \( 10^{251} \equiv 1 \mod 251 \). Thus, (\(^*\)) holds for \( k = 250 \), and so the number \( N_{8,250} = 88 \cdots 8 \) is divisible by 2008.