Math 280 Problems for September 21

Pythagoras Level

Problem 1: The set $S$ contains ten numbers. The mean of the numbers in $S$ is 23. The mean of the six smallest numbers in $S$ is 15. The mean of the six largest numbers in $S$ is 30. What is the median of the numbers in $S$?

[2011NJUMC Ind. #2] Let $x_1, x_2, \ldots, x_{10}$ denote the ten numbers. We are given

\[
x_1 + x_2 + \cdots + x_{10} = 23 \cdot 10 \\
x_1 + x_2 + \cdots + x_6 = 15 \cdot 6 \\
x_5 + x_6 + \cdots + x_{10} = 30 \cdot 6.
\]

Subtracting the first equation from the sum of the other two gives

\[
x_5 + x_6 = 15 \cdot 6 + 30 \cdot 6 - 23 \cdot 10 = 40
\]

Thus the median is $40/2 = 20$.

Problem 2: In the figure below, $A$ and $B$ are the points $(2, 0)$ and $(2, 5)$ respectively ($O$ is the origin). If right triangle $OAB$ is flipped about its hypotenuse as shown, what is the slope of the line through $O$ and $A'$?

[2011NJUMC Ind. # 6] Let $\theta = \angle AOB$. Then the slope of $OA'$ is given by

\[
m = \tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2(5/2)}{1 - (5/2)^2} = -\frac{20}{21}.
\]

Newton Level

Problem 3: Let $f_1(x) = f(x) = \frac{1}{1+2x}$. Then for $n > 1$, left $f_n(x) = f(f_{n-1}(x))$. So, for example, $f_3(x) = f(f(f(x)))$. Compute $f'_n(-1)$.

[2011NJUMC Ind. #4] The important fact to realize here is that $x = -1$ is a fixed point for $f$. In other words, $f(-1) = -1$, and so by induction $f_n(-1) = -1$. Now, by the chain rule, notice that

\[
f'_n(x) = [f(f_{n-1}(x))]' = f'(f_{n-1}(x))f'_{n-1}(x) \\
f'_n(-1) = f'(f_{n-1}(-1))f'_{n-1}(-1) \\
= f'(-1)f'_{n-1}(-1).
\]

In other words, to get the derivative of the next $f_n$ at $x = -1$, we simply multiply the derivative of the previous $f_{n-1}$ at $x = -1$ by the same constant: $f'(-1)$. So we really only need to compute the very first $f'(-1)$, and then the rest of the derivatives will follow easy from the recursive formula. Since

\[
f'(x) = \frac{-2}{(1+2x)^2} \quad \text{and so} \quad f'(-1) = -2
\]

we have $f'_n(-1) = (-2)^n$ and in particular $f'_7(-1) = (-2)^7 = -128$. 
**Problem 4:** Find the limit
\[
\lim_{n \to \infty} \left[ \left( 1 + \frac{1}{n} \right)^n \right]^n.
\]

[2011NJUMC Ind. #13] First we take the natural log of the \( n \)th term, arriving at \( n(n \ln(1 + 1/n) - 1) \). To compute the limit we rewrite
\[
\lim_{n \to \infty} n(n \ln(1 + 1/n) - 1) = \lim_{n \to \infty} \frac{\ln(1 + 1/n) - 1}{1/n^2}.
\]
Using L’Hospital we arrive at a limit of \(-1/2\), so the original limit is \( e^{-1/2} = \frac{1}{\sqrt{e}} \).

**Wiles Level**

**Problem 5:** If \( A \) is the matrix \( \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \), determine the series:
\[
A - \frac{1}{3} A^2 + \frac{1}{9} A^3 + \cdots + \left( -\frac{1}{3} \right)^n A^{n+1} + \cdots
\]

[2011NJUMC Ind. #12] Set the sum equal to \( B \), and multiply both sides by \( I + \frac{1}{3} A \).
\[
(I + \frac{1}{3} A)(A - \frac{1}{3} A^2 + \frac{1}{9} A^3 + \cdots + \left( -\frac{1}{3} \right)^n A^{n+1} + \cdots) = (I + \frac{1}{3} A)B
\]
The left side telescopes and we’re left with \( A = (1 + \frac{1}{3} A)B \). Thus
\[
B = (1 + \frac{1}{3} A)^{-1} B
\]
\[
= 3 \begin{pmatrix} 4 & -3 \\ -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}
\]
\[
= \frac{3}{13} \begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}
\]
\[
= \frac{3}{13} \begin{pmatrix} 1 & -9 \\ -3 & 1 \end{pmatrix}
\]

**Problem 6:** Compute the area of the region which lies between the \( x \)-axis and the curve, \( y = e^{-x} \sin(\pi x) \), for \( x \geq 0 \).

[2011NJUMC Team #5] Integration by parts give us the following anti-derivative for the function.
\[
\int e^{-x} \sin(\pi x) \, dx = \frac{-e^{-x}}{\pi^2 + 1} (\pi \cos(\pi x) + \sin(\pi x)).
\]
Now, we can’t simply use the anti-derivative to evaluate the integral from 0 to 1, because we want area below the \( x \)-axis to count as positive area. So the key is to use the anti-derivative to get a general formula for the integral from \( n \) to \( n + 1 \) of the absolute value.
\[
\int_n^{n+1} |e^{-x} \sin(\pi x)| \, dx = \frac{\pi}{\pi^2 + 1} \left| e^{-n} \cos(\pi n) - e^{-(n+1)} \cos((n+1)\pi) \right|
\]
\[
= \frac{\pi}{\pi^2 + 1} \left( e^{-n} + e^{-(n+1)} \right)
\]
\[
= \frac{\pi(e + 1)}{(\pi^2 + 1)e^{n+1}}
\]
Now we see that the areas over the intervals, \([n, n + 1]\), form a geometric sequence, whose sum is given by
\[
A = \sum_{n=0}^{\infty} \frac{\pi(e + 1)}{(\pi^2 + 1)e^{n+1}} = \frac{\pi(e + 1)}{(\pi^2 + 1)e} \cdot \frac{1}{1 - 1/e} = \frac{\pi(e + 1)}{(\pi^2 + 1)(e - 1)}
\]