Math 280 Problems for September 21

Pythagoras Level

Problem 1: The set S contains ten numbers. The mean of the numbers in S is 23. The mean of the six smallest numbers in S is 15. The mean of the six largest numbers in S is 30. What is the median of the numbers in S?

[2011NJUMC Ind. #2] Let $x_1, x_2, \ldots x_{10}$ denote the ten numbers. We are given

$$x_1 + x_2 + \cdots + x_{10} = 23 \cdot 10$$

$$x_1 + x_2 + \cdots + x_6 = 15 \cdot 6$$

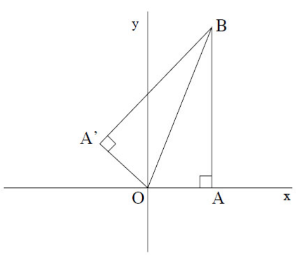
$$x_5 + x_6 + \cdots + x_{10} = 30 \cdot 6.$$

Subtracting the first equation from the sum of the other two gives

$$x_5 + x_6 = 15 \cdot 6 + 30 \cdot 6 - 23 \cdot 10 = 40$$

Thus the median is $40/2 = 20$.

Problem 2: In the figure below, A and B are the points $(2, 0)$ and $(2, 5)$ respectively (O is the origin). If right triangle OAB is flipped about its hypotenuse as shown, what is the slope of the line through O and A'?



[2011NJUMC Ind. # 6] Let $\theta = \angle AOB$. Then the slope of OA' is given by

$$m = \tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2(5/2)}{1 - (5/2)^2} = -\frac{20}{21}.$$

Newton Level

Problem 3: Let $f_1(x) = f(x) = \frac{1}{1+2x}$. Then for $n > 1$, left $f_n(x) = f(f_{n-1}(x))$. So, for example, $f_3(x) = f(f(f(x)))$. Compute $f_7'(-1)$.

[2011NJUMC Ind. #4] The important fact to realize here is that $x = -1$ is a fixed point for f. In other words, $f(-1) = -1$, and so by induction $f_n(-1) = -1$. Now, by the chain rule, notice that

$$\begin{aligned} f_n'(x) &= [f(f_{n-1}(x)]' = f'(f_{n-1}(x))f_{n-1}'(x) \\ f_n'(-1) &= f'(f_{n-1}(-1))f_{n-1}'(-1) \\ &= f'(-1)f_{n-1}'(-1). \end{aligned}$$

In other words, to get the derivative of the next f_n at $x = -1$, we simply multiply the derivative of the previous f_{n-1} at $x = -1$ by the same constant: $f'(-1)$. So we really only need to compute the very first $f'(-1)$, and then the rest of the derivatives will follow easy from the recursive formula. Since

$$f'(x) = \frac{-2}{(1+2x)^2} \text{ and so } f'(-1) = -2$$

we have $f_n'(-1) = (-2)^n$ and in particular $f_7'(-1) = (-2)^7 = -128$.

Problem 4: Find the limit
$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^n}{e}\right]^n.$$

[2011NJUMC Ind. #13] First we take the natural log of the nth term, arriving at $n(n\ln(1+1/n)-1)$. To compute the limit we rewrite
$$\lim_{n\to\infty} n(n\ln(1+1/n)-1) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})-\frac{1}{n}}{\frac{1}{n^2}}.$$
Using L'Hospital we arrive at a limit of $-1/2$, so the original limit is $e^{-1/2}=\frac{1}{\sqrt{e}}$.

Wiles Level

Problem 5: If A is the matrix $\begin{pmatrix}1 & 3\\ -1 & 1\end{pmatrix}$, determine the series:
$$A-\frac{1}{3}A^2+\frac{1}{9}A^3+\cdots+\left(-\frac{1}{3}\right)^n A^{n+1}+\cdots$$

[2011NJUMC Ind. #12] Set the sum equal to B, and multiply both sides by $I+\frac{1}{3}A$.
$$(I+\frac{1}{3}A)(A-\frac{1}{3}A^2+\frac{1}{9}A^3+\cdots+\left(-\frac{1}{3}\right)^n A^{n+1}+\cdots)=(I+\frac{1}{3}A)B$$
The left side telescopes and we're left with $A=(1+\frac{1}{3}A)B$. Thus
$$\begin{aligned}
B &= (1+\frac{1}{3}A)^{-1}B\\
&= 3\begin{pmatrix}4 & -3\\ -1 & 4\end{pmatrix}^{-1}\begin{pmatrix}1 & -3\\ -1 & 1\end{pmatrix}\\
&= \frac{3}{13}\begin{pmatrix}4 & 3\\ 1 & 4\end{pmatrix}\begin{pmatrix}1 & -3\\ -1 & 1\end{pmatrix}\\
&= \frac{3}{13}\begin{pmatrix}1 & -9\\ -3 & 1\end{pmatrix}
\end{aligned}$$

Problem 6: Compute the area of the region which lies between the x-axis and the curve, $y=e^{-x}\sin(\pi x)$, for $x\geq 0$.

[2011NJUMC Team #5] Integration by parts give us the following anti-derivative for the function.
$$\int e^{-x}\sin(\pi x)\ dx = \frac{-e^{-x}}{\pi^2+1}(\pi\cos(\pi x)+\sin(\pi x)).$$
Now, we can't simply use the anti-derivative to evaluate the integral from 0 to 1, because we want area below the x-axis to count as positive area. So the key is to use the anti-derivative to get a general formula for the integral from n to $n+1$ of the absolute value.
$$\begin{aligned}
\int_n^{n+1}|e^{-x}\sin(\pi x)|\ dx &= \frac{\pi}{\pi^2+1}\left|e^{-n}\cos(\pi n)-e^{-(n+1)}\cos((n+1)\pi)\right|\\
&= \frac{\pi}{\pi^2+1}\left(e^{-n}+e^{-(n+1)}\right)\\
&= \frac{\pi(e+1)}{(\pi^2+1)e^{n+1}}
\end{aligned}$$
Now we see that the areas over the intervals, $[n,n+1]$, form a geometric sequence, whose sum is given by
$$A=\sum_{n=0}^{\infty}\frac{\pi(e+1)}{(\pi^2+1)e^{n+1}}=\frac{\pi(e+1)}{(\pi^2+1)e}\cdot\frac{1}{1-1/e}=\frac{\pi(e+1)}{(\pi^2+1)(e-1)}$$