

Math 280 Problems for September 13

Pythagoras Level

Problem 1: The sum of the reciprocals of two real numbers is -1 , and the sum of their cubes is 4. What are the numbers?

Problem 2: Two students play a game based on the total roll of two standard dice. Student A says that a 12 will be rolled first. Student B says that two consecutive 7s will be rolled first. The students keep rolling until one of them wins. What is the probability that A will win?

Newton Level

Problem 3: Evaluate

$$\int_1^{2008} \frac{dx}{x + \lfloor \log_{10} x \rfloor}.$$

(For a real number u , $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u .)

Problem 4: Express the product

$$\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{2008^2}\right)$$

as simply as you can as a rational fraction in lowest terms. Justify your answer.

Wiles Level

Problem 5: Prove that

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Problem 6: For any positive integer n , let $\langle n \rangle$ denote the closest integer to \sqrt{n} . Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$