Math 280 Problems for September 27

Pythagoras Level

Problem 1: Find all pairs of real numbers \((x, y)\) satisfying the system of equations
\[
\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)
\]
\[
\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).
\]

Problem 2: Suppose \(n\) fair 6-sided dice are rolled simultaneously. What is the expected value of the score on the highest valued die?

Newton Level

Problem 3: Let \(f\) be a continuous function on \([0, 1]\), differentiable on \((0, 1)\), and such that \(f(1) = 0\). Show that for some \(c \in (0, 1)\),
\[
\frac{f(c)}{c} = -f'(c).
\]

Problem 4: Let \(f : [0, 1) \to \mathbb{R}\) be a continuous, strictly increasing function, such that
\[
(f(x))^3 = \int_0^x t(f(t))^2 \, dt
\]
for every \(x \geq 0\). Show that for every \(x \geq 0\) we have \(f(x) = \frac{x^2}{6}\).

Wiles Level

Problem 5: Given that \(a\) and \(b\) are real numbers satisfying \(a^3 - 3ab^2 = 39\) and \(b^3 - 3a^2b = \sqrt{487}\), determine \(a^2 + b^2\).

Problem 6: Let \(f\) be a nonconstant polynomial with positive integer coefficients. Prove that if \(n\) is a positive integer, then \(f(n)\) divides \(f(f(n) + 1)\) if and only if \(n = 1\).