Math 280 Problems for October 4

Pythagoras Level

Problem 1: Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer $M$ such that $f(n)$ is divisible by $M$ for every positive integer $n$.

Problem 2: Evaluate

$$\sqrt{\frac{2207}{2207} - \frac{1}{2207-\ldots}}.$$  

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where $a, b, c, d$ are integers.

Hint: $(x^2 - ax + 1)(x^2 + ax + 1) = (x^2)^2 - (a^2 - 2)x^2 + 1$.

Newton Level

Problem 3: Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} \, dx.$$  

Problem 4: Let $n \geq 2$ be an integer and define $f(x) = 1 - x^n$. For each $t \in (0, 1)$, let $A_t$ denote the area of the triangle in the first quadrant formed by the $x$-axis, $y$-axis, and the tangent line to $f(x)$ at $x = t$. Find $t \in (0, 1)$ so that $A_t$ is a minimum.

Wiles Level

Problem 5: Let $n \geq 1$. Pick at random a function

$$f : \{1, \ldots, n\} \to \{1, 2, 3\}$$  

What is the probability $P$ of $f$ not being onto (surjective)?

Problem 6: Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any three (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication.