Math 280 Problems for October 18

Pythagoras Level

Problem 1: Given that
\[ \sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3, \]
compute
\[ r^3 + \frac{1}{r^3}. \]

Problem 2: The positive numbers \( r \) and \( t \) are related by the fact that if the radius \( r \) of a circle is increased by \( t \), the area is doubled. Express \( r \) as a function of \( t \).

Newton Level

Problem 3: Find a real number \( r \) such that if an integer \( n \geq r \), then
\[ \frac{1}{n} + \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{n^2} \geq 2008. \]

Problem 4: Let \( f(x) = \int_0^x e^{-t^2} \, dt \). Given that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \), evaluate
\[ \int_0^{\infty} e^{-x^2 + f(x)} \, dx. \]

Wiles Level

Problem 5: A card shuffling machine always rearranges cards in the same way relative to the order in which they are given to it. The thirteen spades arranged in the order

\[ A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K \]

are put into the machine, shuffled, and then the shuffled cards are put into the machine and shuffled again. If at this point the order of the cards is

\[ 3, K, 10, 2, Q, 9, 4, J, 8, 6, 7, A, 5, \]

what was the order of the cards after the first shuffle?

Problem 6: For positive numbers \( r \), let \( F(r) \) denote the fractional part of \( r \); i.e., \( F(r) = r - \lfloor r \rfloor \). Thus, e.g., \( F(8/3) = 2/3 \). Find a positive number \( r \) such that
\[ F(r) + F\left(1/r\right) = 1. \]