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One way to understand $\lim_{n \to \infty} (1 + \frac{1}{n})^n$

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About e			

$e = 2.718281828 \cdots$ is a mathematical constant called **Euler's** number after the Swiss mathematician Leonhard Euler.

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Definitions o	of e		

Most commonly, we define e as

Definition

$$e \equiv \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

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Definitions of	fe		

Most commonly, we define e as

Definition

$$e \equiv \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Definition

$$e\equiv\sum_{n=0}^{\infty}\frac{1}{n!}$$

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Visualization

Take a look at the following graph



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Question			

Why is e defined as
$$\lim_{n\to\infty}(1+\frac{1}{n})^n$$
, not anything else?

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Recall			

Let's first recall the definition of the derivative:

Definition

The **derivative** of f(x) with respect to x is the function f'(x) and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Ideas			
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Consider the derivative of every exponential function of the form $f(x) = a^x$ (a > 0).

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Consider the derivative of every exponential function of the form $f(x) = a^x$ (a > 0).

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$
$$= a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h}\right)$$

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Ideas			

Consider the derivative of every exponential function of the form $f(x) = a^x$ (a > 0).

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$
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Notice here that the derivative of f(x) is equal to a multiple of itself.

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Computation			

• To obtain a numerical value of *a*, we let $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$.

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Computation

- To obtain a numerical value of a, we let $\lim_{h \to 0} \frac{a^h 1}{h} = 1$.
- So for small values of *h*, we can write:

$$a^h - 1 \approx h$$

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Computation

- To obtain a numerical value of a, we let $\lim_{h \to 0} \frac{a^h 1}{h} = 1$.
- So for small values of *h*, we can write:

$$a^h - 1 pprox h \implies a^h pprox 1 + h$$

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Computation

- To obtain a numerical value of *a*, we let $\lim_{h\to 0} \frac{a^h 1}{h} = 1$.
- So for small values of *h*, we can write:

$$a^h-1pprox h\implies a^hpprox 1+h\implies approx (1+h)^{rac{1}{h}}.$$

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We are almost there

If we replace
$$h$$
 by $\frac{1}{n}$, then $a pprox (1+\frac{1}{n})^n.$

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Finally			

The approximation gets better as n gets larger, then

$$a = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

so that
$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

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Finally			

The approximation gets better as n gets larger, then

$$a = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

so that
$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$
 (proportional constant).

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Recall			

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$
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$$= a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h}\right)$$

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Recall			

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$
$$= a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h}\right)$$

Thus,

$$a = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \implies f'(x) = a^x \cdot 1 = a^x.$$

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Why not give a a new name since it is a constant? Call it e!

$$e\equiv\lim_{n\to\infty}\left(1+rac{1}{n}
ight)^n.$$

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Thanks fo	sr lictoning		

Thanks for listening!

Questions?

Le Tang One way to understand $\lim_{n\to\infty} (1+\frac{1}{n})^n$

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